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# Value at Risk for Integrated Returns and Its Applications to Equity Portfolios

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**Abstract**

The present paper investigates the distribution quantile for integrated portfolio returns that follow a general class of multivariate stochastic volatility model. We propose a non-parametric quantile estimate that incorporates the rate with which the true quantile diverges as the integration horizon expands. The asymptotic normality established for the estimate enables us to construct the confidence interval for the true quantile. Monte Carlo experiments are conducted to demonstrate both the consistency and the advantages of our approach. Results on quantile estimates for the return distribution of the S&P 500 index are also presented.

Keywords: Quantile; Value at Risk; Stochastic Volatility Model; Integrated returns

## 1 Introduction

Representing the quantile of a portfolio's return distribution, Value at Risk (VaR) is a quantitative measure developed to deal with market risk that is, like credit risk and liquidity risk, recognized as an important aspect of financial risk (Jorion (2007)). Despite the criticism that VaR does not fall into the category of coherent measures (Artzner et al. (1999)), VaR has been widely used by practitioners in financial industries. This popularity is in part due to the fact that the methodology for the computation and statistical analysis of VaR are well established. In the abundant literature on VaR, a frequently used assumption is that the return sequence has a stationary distribution. Very few works have looked into the non-stationary case. Perhaps this is because finding the time-varying quantile of a non-stationary distribution does not appear to be a well-posed problem, at least from the estimation point of view, when the source of non-stationarity is not deterministic. The purpose of this article is to evaluate VaR under a typical circumstance of non-stationarity that the underlying distribution is based on an integrated process with incremental returns belonging to a class of non-linear models.

There are two concerns that motivate the present study. First, large institutional investors such as insurance companies, pension funds, and sovereign funds usually set a very long investment horizon for their portfolios, which could last for several years depending on the nature of the funds. In order to assess the market risk to which the portfolio's positions are exposed, VaR is one of the popular measures that fund managers or financial advisors employ to quantify the risk. Since the equity's holding period is of long duration, it is appropriate to work on the distribution derived from the portfolio's integrated returns to evaluate the VaR. Second, these long-term investors often need some financial vehicles to hedge the risk incurred by the changes of currency rates or interest rates. The instruments frequently used are either some OTC warrant-type contracts or publicly traded equity options such as the LEAPS (Long-term Equity Anticipation Securities) issued by CBOE (Chicago Board Options Exchange). The maturity of the latter is from one to three years and that of the former could be even longer. Furthermore, because of the increasing demand for longer-term hedging tools for equities by institutional investors after the global financial crisis in 2008, CBOE launched in March 2012 the Super LEAPS option (an European-style option on S&P 500 index) with maturity of five years. For the seller of these contracts, one of the major concerns is a large swing of the equity price away from the strike price (or the spot price near at-the-money at the transaction time), which can be quantitatively measured by the VaR of the equity's integrated returns.

In view of the nonlinear nature of equity returns (Taylor (1986)), two popular classes of stationary time series models have been proposed to describe the return dynamics: the ARCH (or GARCH) family (Engle (1982); Bollerslev (1986)) and the stochastic volatility (SV) model (Taylor (1986)). Both are able to capture some principal stylized facts exhibited by speculative equity returns such as volatility clustering (Mandelbrot (1963)) and Taylor's effect (Taylor (1986)). We focus on the SV model to cover the popular regime-switching log-normal model (Hamilton (1994); Hardy (2001); Hardy, Freeland, and Till (2006)).

To evaluate the quantiles of return distribution derived by using the whole duration, the standard approach in the literature is to use the equity's past returns of some fixed frequency (daily or weekly for example) to identify the parametric model, GARCH or SV, chosen in advance. Then use the model with estimated parameters to simulate a large number of price paths with the given duration. The collection of all the simulated prices in turn gives the empirical distribution of the return and from which the quantile of interest is determined (Hardy (2001); Hardy, Freeland, and Till (2006)). The main concern here is the lack of analytical guidance to address the issue of inference.

To fill the gap, we employ a more direct approach that consists of two parts. The first is to propose an estimate of the quantile for the underlying integrated process. Second, we derive a closed-form formula for the quantile's true value, and show that the central limit theorem for the deviation between the estimate and the true value holds. Before we formally present the results, it would be helpful to explain our method heuristically. To simplify the issue, let $\{y_t = a_t u_t,\ t = 1, \ldots, T\}$ be an observed sequence of independent returns with volatility changing over time, where each $y_t$ is normally distributed with mean zero and non-random variance $a_t^2 > 0$. Suppose $\sigma_T^2 \equiv T^{-1} \sum_{t=1}^T a_t^2$ converges to $\sigma_a^2 > 0$ as $T$ tends to infinity. We are concerned with the $\alpha$-th VaR $q_\alpha$ of the integrated returns $Y_T = \sum_{t=1}^T y_t$ with horizon $T$, finding $q_\alpha$ which satisfies $\alpha = P(Y_T < q_\alpha)$ or, equivalently,

$$\alpha = P\left(N(0,1) < q_\alpha/(\sqrt{T}\sigma_T)\right),$$

implying that

$$q_\alpha = \Phi^{-1}(\alpha)\sqrt{T}\sigma_T \approx \Phi^{-1}(\alpha)\sqrt{T}\sigma_a, \tag{1}$$

where the approximation is justified by $\lim_{T\to\infty} \sigma_T^2 = \sigma_a^2$. Because $\lim_{T\to\infty} T^{-1} \sum_t^T y_t^2 = \sigma_a^2$ with probability one, we can estimate $q_\alpha$ by

$$\hat{q}_\alpha = \Phi^{-1}(\alpha)\sqrt{T}\hat{\sigma}_a \tag{2}$$

with $\hat{\sigma}_a^2 = \sum_t^T y_t^2/T$. Assuming further that

$$\sigma_T^2 = \sigma_a^2 + o(T^{-1/2}) \quad \text{as} \quad T \to \infty, \tag{3}$$

we have, by the Central Limit Theorem,

$$\hat{q}_\alpha - q_\alpha = \Phi^{-1}(\alpha)\sqrt{T}\,(\hat{\sigma}_a - \sigma_a) \xrightarrow{d} N(0, c^2), \tag{4}$$

where $c^2 = a^*/(2\sigma_a^2)$ with $a^* = \lim_{T\to\infty} T^{-1} \sum_{t=1}^T a_t^4$. Thus, although the distribution of $S_T$ is non-stationary, its $\alpha$-th quantile diverges with an explicit rate that can be estimated. As a result, one can construct confidence intervals by using (4) to infer the estimation error of (2).

Our technical task is then to extend the derivation of (1) and (4) to the SV model where $\{a_t\}$ is the exponential transformation of a stationary linear process independent of $\{u_t\}$. Results of the extension are stated in Theorems 1 and 2 of Section 2 where a multivariate version of the SV process is adopted to allow modeling flexibility. In Section 3 we examine, through simulation, the finite sample performance of the VaR estimate we propose, and demonstrate that our approach is robust against non-normal $\{u_t\}$, and is superior to the traditional simulation-based method in terms of coverage ratios for confidence intervals. Section 4 presents results on quantile estimates for the return distribution of the S&P 500 index. Proofs of theorems are in the online supplementary document.

# 2 Multivariate Stochastic Volatility Model for Equity Portfolios

We are concerned with the Value-at-Risk (VaR) of the integrated returns of a portfolio consisting of $m$ weighted component assets. The return model for the assets is as follows. Let $r_t = (r_{1,t}, ..., r_{m,t})'$ be the equity return vector at time $t$. The multivariate SV (MSV) model is

$$r_t = \mu + V_t U_t, \tag{5}$$

where $\mu = (\mu_1, ..., \mu_m)'$ is the mean of $r_t$, $V_t = \text{diag}(v_{1,t}, ..., v_{m,t})$, a diagonal matrix, where $v_{i,t} = \exp(Z_{i,t}/2)$, $i = 1, ..., m$, $U_t = (u_{1,t}, ..., u_{m,t})'$ is a sequence of shocks comprised of independent identically distributed (i.i.d.) random vectors with mean 0 and a positive-definite covariance matrix $\Sigma_U = [\sigma_{U,ij}]$, and $Z_t = (Z_{1,t}, ..., Z_{m,t})'$ is an $m$-dimensional stationary short-memory process. Here, $\{Z_t\}$ is an $m$-dimensional linear process

$$Z_t = \mu_z + \sum_{s=0}^{\infty} A_s \eta_{t-s}, \tag{6}$$

where $\mu_z = (\mu_{z,1}, ..., \mu_{z,m})$, $A_s = [A_{ij}^{(s)}]$, $\eta_t = (\eta_{1,t}, ..., \eta_{m,t})'$, $\{\eta_t\}$ is a sequence of i.i.d. random vectors with 0 mean, a positive-definite covariance matrix $\Sigma_\eta = [\sigma_{\eta,ij}]$, and is independent of $\{U_t\}$. Short-memory of $\{Z_t\}$ requires

$$\sum_{s=0}^{\infty} \left| A_{ij}^{(s)} \right| < \infty, \quad \text{for} \quad i = 1, ..., m, \quad \text{and} \quad j = 1, ..., m.$$

See Chapter 10 of Hamilton (1994) for further discussion. The MSV model we adopt is similar to that of Harvey, Ruiz, and Shephard (1994) where $\{Z_t\}$ is a vector autoregressive model of order 1 with a diagonal coefficient matrix, while in our model, $\{Z_t\}$ is the more general linear process at (6). The model of Harvey et al. (1994) is the first MSV model proposed in the literature. Since then, a wide range of MSV models has been developed. See, for example, Asai, McAleer, and Yu (2006) and Yu and Meyer (2006), and references therein.

For a given set of weights $\{w_i : 1 \le i \le m\}$ satisfying $w_i > 0$ and $\sum_{i=1}^m w_i = 1$, let $\tilde{r}_t$ be the return of the weighted portfolio,

$$\tilde{r}_t = \sum_{i=1}^{m} w_i r_{i,t} = \tilde{\mu} + \sum_{i=1}^{m} w_i v_{i,t} u_{i,t}, \tag{7}$$

with mean $\tilde{\mu} = \sum_{i=1}^m w_i \mu_i$ and variance $\sigma^2 = E(\tilde{r}_t - \tilde{\mu})^2$. We first establish a central limit theorem for $\hat{\mu}_T = T^{-1} \sum_{t=1}^T \tilde{r}_t$ and $\hat{\sigma}_T^2 = T^{-1} \sum_{t=1}^T (\tilde{r}_t - \hat{\mu}_T)^2$ derived from the return sequence $\{\tilde{r}_t : t = 1, \ldots, T\}$ following the SV model defined in (5), (6), and (7).

**Theorem 1** *Assume that the portfolio returns $\{\tilde{r}_t\}$ follow the SV model specified at (7) such that, for each $i$, $Ev_{1,1}^4 < \infty$ and $\{u_{i,t}\}$ is a sequence of i.i.d. mean-zero-unit-variance random variables independent of $\{V_t\}$. Then as $T \to \infty$*

$$\sqrt{T}\,(\hat{\mu}_T - \tilde{\mu}) \xrightarrow{d} N(0, \sigma^2) \tag{8}$$

$$\sqrt{T}\left(\hat{\sigma}_T^2-\sigma^2\right)\xrightarrow{d}N\left(0,g^2\right). \tag{9}$$

*If* $\{\eta_t\}\sim N(0,\Sigma_\eta)$, *then*

$$\begin{aligned}
g^2 &= \sum_{i,j,k,l=1}^{m} w_iw_jw_kw_l\left(\sigma_{U,ik}\sigma_{U,jl}+\sigma_{U,il}\sigma_{U,jk}\right)e^{J_4'\mu_z(i,j,k,l)+\frac{1}{2}J_4'\Sigma_Z(i,j,k,l)J_4}\\
&\quad+\sum_{i,j,k,l=1}^{m} w_iw_jw_kw_l e^{J_4'\mu_z(i,j,k,l)+\frac{1}{2}J_2'\{\Sigma_Z(i,j)+\Sigma_Z(k,l)\}J_2}\sigma_{U,ij}\sigma_{U,kl}\left(e^{\sigma_{Z,ik}+\sigma_{Z,il}+\sigma_{Z,jk}+\sigma_{Z,jl}}-1\right)\\
&\quad\times\left\{1+\frac{2}{e^{\sigma_{Z,ik}+\sigma_{Z,il}+\sigma_{Z,jk}+\sigma_{Z,jl}}-1}\sum_{u=1}^{\infty}\left(e^{J_2'\{\Sigma_{Z,\{(i,j),(k,l)\}}(-u)\}J_2}-1\right)\right\},
\end{aligned}\tag{10}$$

*where* $J_2=(1,1)'$, $J_4=(1,1,1,1)'$, $\mu_z(i,j,k,l)=(\mu_{z,i},\mu_{z,j},\mu_{z,k},\mu_{z,l})'$,

$$\Sigma_Z(i,j)=\begin{pmatrix}\sigma_{Z,ii} & \sigma_{Z,ij}\\ \sigma_{Z,ji} & \sigma_{Z,jj}\end{pmatrix},$$

$$\Sigma_Z(i,j,k,l)=\begin{pmatrix}\sigma_{Z,ii} & \sigma_{Z,ij} & \sigma_{Z,ik} & \sigma_{Z,il}\\ \sigma_{Z,ji} & \sigma_{Z,jj} & \sigma_{Z,jk} & \sigma_{Z,jl}\\ \sigma_{Z,ki} & \sigma_{Z,kj} & \sigma_{Z,kk} & \sigma_{Z,kl}\\ \sigma_{Z,li} & \sigma_{Z,lj} & \sigma_{Z,lk} & \sigma_{Z,ll}\end{pmatrix},$$

$$\Sigma_Z=[\sigma_{Z,ij}]=E\left[(Z_t-\mu_z)(Z_t-\mu_z)'\right],$$

$$\Sigma_Z(r)=[\sigma_{Z,ij}(r)]=E\left[(Z_t-\mu_z)(Z_{t-r}-\mu_z)'\right],$$

*and*

$$\Sigma_{Z,\{(i,j),(k,l)\}}(-u)=\begin{pmatrix}\sigma_{Z,ik}(-u) & \sigma_{Z,il}(-u)\\ \sigma_{Z,jk}(-u) & \sigma_{Z,jl}(-u)\end{pmatrix}.$$

Let $\{S_t,t=0,1,\ldots,T\}$ be the price process formed by the portfolio returns $\{\tilde{r}_t\}$, $\ln(S_T/S_0)=\sum_{t=1}^T\tilde{r}_t$. Denote by $Q_\alpha(T)$ the $\alpha$-th quantile of $S_T$ and set $A_T=(\ln(Q_\alpha(T)/S_0)-T\tilde{\mu})/\sqrt{T}$. Thus,

$$\alpha=P\left(\frac{\sum_{t=1}^T\tilde{r}_t-T\tilde{\mu}}{\sqrt{T}\sigma}<\frac{A_T}{\sigma}\right), \tag{11}$$

with $\sigma^2=E(\tilde{r}_t-\tilde{\mu})^2$. For (11) to hold, one needs the distribution of $\tilde{r}_t$ to be continuous. This follows from the normality assumtion of $U_t$ imposed later in Theorem 2. From the central limit theorem established in (8), we have $A_T/\sigma\approx\Phi^{-1}(\alpha)$, which suggests an estimate for $Q_\alpha(T)$,

$$\hat{Q}_\alpha(T)=S_0\exp\left\{\Phi^{-1}(\alpha)\sqrt{T}\hat{\sigma}_T+T\mu^*\right\}, \tag{12}$$

where $\hat{\sigma}_T = (T^{-1}\sum_{t=1}^{T}(\tilde{r}_t - \mu^*)^2)^{1/2}$ with $\mu^* = \tilde{\mu}$ if $\tilde{\mu}$ is known or $\mu^* = \hat{\mu}_T = T^{-1}\sum_{t=1}^{T}\tilde{r}_t$ otherwise.

The next result gives the asymptotical normality of the sequence of estimation errors $\ln(\hat{Q}_\alpha(T)/Q_\alpha(T))$.

**Theorem 2** *Under the assumptions of Theorem 1 and the normality of $U_t$, if $\tilde{\mu}$ is known,*

$$\ln\left(\hat{Q}_\alpha(T)/Q_\alpha(T)\right) \xrightarrow{d} N(0, (g\Phi^{-1}(\alpha)/(2\sigma))^2) \tag{13}$$

*with same $g^2$ in (10); if $\tilde{\mu}$ is unknown,*

$$T^{-1/2}\ln\left(\hat{Q}_\alpha(T)/Q_\alpha(T)\right) \xrightarrow{d} N(0, \sigma^2). \tag{14}$$

Theorems 1 and 2 still hold if the weights $\{\omega_i\}$ are allowed to be negative, mainly because the central limit theorem developed in Theorem 1 is not affected by this. Theorem 2 has the implication that, although the quantile $Q_\alpha(T)$ diverges as $T \to \infty$, one can still assess its asymptotical location by using the confidence intervals derived from (13) or (14). In contrast to the usual cases, the width of confidence interval formed by (13) or (14) does not shrink as $T \to \infty$. This is expected since the parameter (quantile or VaR) of interest belongs to the distribution of a sum of stationary variables instead of the common distribution of the variables themselves.

# 3 Simulation Studies

## 3.1 Empirical Coverage Probabilities of the Confidence Intervals

From Section 2, for $\tilde{\mu}$ known, the $100(1-\beta)\,\%$ confidence interval for $Q_\alpha(T)$ or the $100\alpha\%$ quantile of liability distribution is

$$\hat{Q}_\alpha(T)\exp(-hU) \le Q_\alpha(T) \le \hat{Q}_\alpha(T)\exp(-hL), \tag{15}$$

where $L$ and $U$ are the $100\,(\beta/2)\,\%$ and the $100\,(1-\beta/2)\,\%$ standard normal quantiles, respectively, and $h = \left\{(g\Phi^{-1}(\alpha)/(2\sigma))^2\right\}^{1/2}$, where $g^2$ is at (10), and $\Phi^{-1}(\cdot)$

is the inverse distribution function of the standard normal. For $\tilde{\mu}$ unknown, the $100(1-\beta)\,\%$ confidence interval for $V_\alpha(T)$ is

$$\hat{Q}_\alpha(T)\exp\left(-\sqrt{T}\hat{\sigma}_T U\right) \le Q_\alpha(T) \le \hat{Q}_\alpha(T)\exp\left(-\sqrt{T}\hat{\sigma}_T L\right). \qquad (16)$$

Here (12), (15), and (16) give illustrations of how Theorems 1 and 2 are used to obtain point and interval estimators for $Q_\alpha(T)$ when the objective function is a monotone function of $S_T$ . The results are not applicable if the objective function is not a monotone (or piecewise monotone) function of $S_T$. For more general payoff functions, it is a challenging problem if the function form is unknown. In the context of regression estimation, Park and Phillips (1999, 2001) are among the few works published on nonlinear transformations of integrated returns.

To form the confidence interval specified in (15), one needs a good estimate for $g^2$. Following (10) to directly estimate $g^2$ may not be feasible, since the autocorrelation functions of $V_t$ and $U_t$, are difficult to separate in the multivariate case. Instead, we employ a resampling scheme, the sampling window method, that has been established for dependent data; see Politis, Romano, and Wolf (1999) for a comprehensive survey on the topic and references therein. Because $g^2$ is the long-run variance of $\sqrt{T}(\hat{\sigma}_T^2 - \sigma^2)$, it suffices to focus on the variance derived from subsamples. Specifically, let $B_i = \left(r_i, ..., r_{i+b-1}\right)$ denote the $i$-th subsample of block size $b$, $1 \le i \le T-b+1$, and $\hat{\mu}_i = b^{-1}\sum_{t=i}^{i+b-1} r_t$. By using the sample variance of $B_i$, $\hat{\sigma}_{T,b,i}^2 = \sum\limits_{t=i}^{i+b-1}\left(r_t - \hat{\mu}_i\right)^2/(b-1)$, we consider as an estimator for $g^2$,

$$\frac{1}{T-b+1}\sum_{i=1}^{T-b+1}\left\{\sqrt{b}(\hat{\sigma}_{T,b,i}^2 - \hat{\sigma}_T^2)\right\}^2, \qquad (17)$$

where $b = cT^{1/3}$ for some $c \ge 1$.

We conducted numerical studies to investigate the empirical coverage rates of the confidence intervals (15) and (16). We considered $m=1$, $m=2$ and $m=10$. For $m=1$, we simulated $\{r_t = r_{1,t}\}_{t=1,\ldots,T}$ from

$$r_t = \mu + V_t U_t, \quad V_t = \bar{\sigma}\exp(Z_t/2), \qquad (18)$$

where $\{Z_t = Z_{1,t}\}$ is the Gaussian AR(1) process determined by $Z_t = \phi Z_{t-1} + \epsilon_t$, and $\varepsilon_t \overset{i.i.d.}{\sim} N\left(0, \bar{\beta}^2\left(1-\phi^2\right)\right)$. We took $S_0 = 1$, $\mu = 0.0003$, $\bar{\sigma} = 0.0099$, $\bar{\beta} = 0.4$, $\alpha = 0.95$ and $0.99$, $\phi = 0.1$, $0.3$, $0.5$, $0.7$, and $0.9$, and $T =$ 2,500 and 5,000.

For $\{U_t = u_{1,t}\}$, we took the distributions for $U_t$ as standard normal; generalized error distribution (GED) with mean=0, sd=1, $\nu = 1$, 1.5, and 2 (for $\nu = 2$, it is just standard normal); skew-normal (SN) with $(\xi, \omega, \alpha, \tau) = (1.22, 1.58, -4, 0)$, $(0.68, 1.21, -1, 0)$, $(0, 1, 0, 0)$, $(-0.68, 1.21, 1, 0)$, and $(-1.22, 1.58, 4, 0)$. (The values of $\xi$ and $\omega$ ensure that the mean and variance of $U_t$ are 0 and 1, respectively). The purpose of considering non-normal distributions is to highlight the robustness of our approach against departures from normality.

For $m = 2$ and $m = 10$, we considered $r_t = \mu + \bar{\sigma} V_t U_t$, $Z_t = \Phi Z_{t-1} + \epsilon_t$, and

$$\begin{pmatrix} U_t \\ \epsilon_t \end{pmatrix} \overset{i.i.d.}{\sim} MVN_m \left( \begin{pmatrix} 0_m \\ 0_m \end{pmatrix}, \begin{pmatrix} \Sigma_U & 0_{m,m} \\ 0_{m,m} & \Sigma_\epsilon \end{pmatrix} \right),$$

where $MVN_m$ stands for $m$-dimensional multivariate normal, $\mu = (0.0003, ..., 0.0003)$, $\Phi = \phi I_m$, $I_m$ is the $m \times m$ identity matrix, $V_t = \text{diag}(v_{1,t}, ..., v_{m,t})$, $v_{i,t} = \exp(Z_{i,t}/2)$, $i = 1, ..., m$, $0_m$ is the $m \times 1$ zero vector, and $0_{m,m}$ is the $m \times m$ zero matrix. We took $w_i = 1/m$, for all $i$, and the values of $\bar{\sigma}$, $\phi$, $\alpha$, and $T$ to be the same as those of the univariate case.

For $m = 2$,

$$\Sigma_U = \begin{bmatrix} 1 & \rho_u \\ \rho_u & 1 \end{bmatrix}, \quad \Sigma_\epsilon = \bar{c} \begin{bmatrix} 1 & \rho_\epsilon \\ \rho_\epsilon & 1 \end{bmatrix},$$

where $\bar{c} = \bar{\beta}^2(1 - \phi^2)$, and again $\bar{\beta} = 0.4$. For $(\rho_u, \rho_\epsilon)$, we considered $(\rho_u, \rho_\epsilon) = (0, 0)$, $(\rho_u, \rho_\epsilon) = (-0.5, 0.5)$, $(\rho_u, \rho_\epsilon) = (0.5, 0.5)$, and $(\rho_u, \rho_\epsilon) = (-0.5, -0.5)$.

For $m = 10$, $\Sigma_U = [\Sigma_U(i, j)]$, where

$$\Sigma_U(i, j) = \begin{cases} 1 & \text{if } i = j, \\ \rho_{U,1} & \text{if } i \neq j, i \wedge j = 1, (i - j) \text{ is odd}, \\ \rho_{U,2} & \text{if } i \neq j, i \wedge j = 1, (i - j) \text{ is even}, \\ \rho_{U,3} & \text{if } i \neq j, i \wedge j \geq 2, (i - j) \text{ is odd}, \\ \rho_{U,4} & \text{if } i \neq j, i \wedge j \geq 2, (i - j) \text{ is even}, \end{cases}$$

$i \wedge j = \min(i, j)$, and $\Sigma_\epsilon$ is defined similarly. For $\rho_U = (\rho_{U,1}, ..., \rho_{U,4})$ and $\rho_\epsilon = (\rho_{\epsilon,1}, ..., \rho_{\epsilon,4})$, we considered $(\rho_U, \rho_\epsilon) = (\tilde{a}, \tilde{a})$, $(\rho_U, \rho_\epsilon) = (\tilde{b}, \tilde{b})$, $(\rho_U, \rho_\epsilon) = (\tilde{c}, \tilde{c})$, $(\rho_U, \rho_\epsilon) = (\tilde{c}, \tilde{b})$, and $(\rho_U, \rho_\epsilon) = (\tilde{b}, \tilde{c})$, where $\tilde{a} = (0, ..., 0)$, $\tilde{b} = (0.5, 0.5, 0.25, 0.25)$, and $\tilde{c} = (-0.5, 0.5, -0.25, 0.25)$.

The coverage probabilities of 95% confidence intervals for $Q_\alpha(T)$ based on stochastic volatility sequences for $(T, \alpha) = (2{,}500, 0.95)$, $(5{,}000, 0.95)$, $(2{,}500, 0.99)$, and $(5{,}000, 0.99)$ are summarized in Tables 1, 2, 3, and 4, respectively. The true $Q_\alpha(T)$ for each of the cases was computed based on $10^6$ price paths with the given $(T, \alpha)$, modeling distribution and parameters, 10,000 replicates were used to calculate the probabilities. For the case of known mean, $c$ was set to be 3 for the block size when the sampling window method was applied to estimate $g^2$. In Tables 1, 2, 3, and 4, the smallest and largest coverage probabilities are 0.9053, and 0.9587, respectively. The results in the tables show that the empirical coverage probabilities are all close to their nominal counterparts. In general, the empirical coverage probabilities when $\mu$ unknown are closer to the nominal counterparts than those when $\mu$ is known.

## 3.2 Comparison with the simulation-based Method

We conducted numerical experiments to compare the empirical coverage probabilities of our non-parametric method with the traditional simulation-based method. For the ease of simulation, we focused on the univariate case. For the traditional simulation-based method, we used the following steps to get the coverage probabilities of 95% confidence intervals for $Q_\alpha(T)$.

Step 1. Simulate $\{r_t = r_{1,t}\}_{t=1,\dots,T}$ from the true data generating process.

Step 2. Estimate the parameters. In estimating the SV models, three methods are available: the method of moment, the quasi-maximum likelihood (QML) approach of Harvey, Ruiz, and Shephard (1994) and the Bayesian approach of Jacquier, Polson, and Rossi (1994). As documented in Table 4 of Jacquier, Polson, and Rossi (1994), the Bayesian approach is superior to the other two methods in terms of bias and standard error, we adopt the Bayesian approach. (The Bayesian approach is implemented by the R package 'stochvol', available at http://cran.r-project.org/web/packages/stochvol/index.html.)

Step 3. Simulate $\{r_t\}$ from the data generating process with the parameter values estimated from Step 2.

Step 4. Compute $H = S_0 \exp(\sum_{t=1}^{T} r_t)$ based on the $\{r_t\}$ generated from Step 3.

Step 5. Repeate Steps 3 - 4 1,000 times to get $H_1$, ..., $H_{1{,}000}$.

Table 1: Coverage probabilities of 95% confidence interval for $Q_\alpha(T)$ based on stochastic volatility sequences. The results are based on 10,000 replicates, and the true $Q_\alpha(T)$ computed by simulating $10^6$ price paths from the true model, $T = 2{,}500$, and $\alpha = 0.95$.

| $\mu$ | known | | | | | unknown | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $U_t \setminus \phi$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
| (1) | 0.9499 | 0.9445 | 0.9445 | 0.9449 | 0.9253 | 0.9506 | 0.9479 | 0.9503 | 0.9498 | 0.9493 |
| (2) | 0.9464 | 0.9421 | 0.9385 | 0.9416 | 0.9340 | 0.9471 | 0.9501 | 0.9506 | 0.9530 | 0.9492 |
| (3) | 0.9495 | 0.9461 | 0.9406 | 0.9456 | 0.9272 | 0.9513 | 0.9514 | 0.9489 | 0.9465 | 0.9535 |
| (4) | 0.9463 | 0.9499 | 0.9479 | 0.9432 | 0.9275 | 0.9501 | 0.9551 | 0.9441 | 0.9473 | 0.9484 |
| (5) | 0.9468 | 0.9441 | 0.9480 | 0.9445 | 0.9257 | 0.9478 | 0.9484 | 0.9509 | 0.9493 | 0.9472 |
| (6) | 0.9386 | 0.9412 | 0.9415 | 0.9340 | 0.9233 | 0.9480 | 0.9446 | 0.9461 | 0.9475 | 0.9456 |
| (7) | 0.9467 | 0.9449 | 0.9450 | 0.9459 | 0.9298 | 0.9536 | 0.9522 | 0.9573 | 0.9568 | 0.9541 |
| (8) | 0.9451 | 0.9437 | 0.9438 | 0.9390 | 0.9271 | 0.9493 | 0.9484 | 0.9482 | 0.9506 | 0.9470 |
| (9) | 0.9422 | 0.9449 | 0.9481 | 0.9424 | 0.9247 | 0.9520 | 0.9499 | 0.9515 | 0.9498 | 0.9475 |
| (10) | 0.9478 | 0.9457 | 0.9462 | 0.9419 | 0.9381 | 0.9492 | 0.9497 | 0.9475 | 0.9504 | 0.9482 |
| (11) | 0.9461 | 0.9391 | 0.9444 | 0.9404 | 0.9241 | 0.9459 | 0.9483 | 0.9535 | 0.9489 | 0.9496 |
| (12) | 0.9451 | 0.9450 | 0.9426 | 0.9398 | 0.9271 | 0.9514 | 0.9507 | 0.9495 | 0.9511 | 0.9478 |
| (13) | 0.9490 | 0.9462 | 0.9467 | 0.9436 | 0.9408 | 0.9508 | 0.9499 | 0.9464 | 0.9495 | 0.9516 |
| (14) | 0.9470 | 0.9466 | 0.9478 | 0.9469 | 0.9432 | 0.9511 | 0.9513 | 0.9518 | 0.9514 | 0.9488 |
| (15) | 0.9476 | 0.9487 | 0.9477 | 0.9473 | 0.9391 | 0.9510 | 0.9527 | 0.9519 | 0.9529 | 0.9514 |
| (16) | 0.9501 | 0.9522 | 0.9501 | 0.9503 | 0.9439 | 0.9477 | 0.9494 | 0.9492 | 0.9506 | 0.9506 |
| (17) | 0.9481 | 0.9470 | 0.9495 | 0.9519 | 0.9432 | 0.9533 | 0.9530 | 0.9518 | 0.9531 | 0.9527 |
| (18) | 0.9479 | 0.9471 | 0.9492 | 0.9480 | 0.9322 | 0.9526 | 0.9518 | 0.9518 | 0.9514 | 0.9490 |

(1) standard normal; (2) GED(0,1,1); (3) GED(0,1,1.5); (4) GED(0,1,2); (5) SN(-0.68,1.21,1); (6) SN(-1.22,1.58,4); (7) SN(1.22,1.58,-4); (8) SN(0.68,1.21,-1); (9) SN(0,1,0,0); (10) $\text{MVN}_2$(0,0); (11) $\text{MVN}_2$(-0.5,0.5); (12) $\text{MVN}_2$(0.5,0.5); (13) $\text{MVN}_2$(-0.5,-0.5); (14) $\text{MVN}_{10}(\tilde{a},\tilde{a})$; (15) $\text{MVN}_{10}(\tilde{b},\tilde{b})$; (16) $\text{MVN}_{10}(\tilde{c},\tilde{c})$; (17) $\text{MVN}_{10}(\tilde{c},\tilde{b})$; (18) $\text{MVN}_{10}(\tilde{b},\tilde{c})$, where $\tilde{a} = (0, ..., 0)$, $\tilde{b} = (0.5, ..., 0.5)$, and $\tilde{c} = (-0.5, 0.5, -0.5, 0.5)$.

Table 2: Coverage probabilities of 95% confidence interval for $Q_\alpha(T)$ based on stochastic volatility sequences. The results are based on 10,000 replicates, and the true $Q_\alpha(T)$ computed by simulating $10^6$ price paths from the true model, $T = 5{,}000$, and $\alpha = 0.95$.

| $\mu$ | known | | | | | unknown | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $U_t \setminus \phi$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
| (1) | 0.9461 | 0.9480 | 0.9511 | 0.9478 | 0.9318 | 0.9487 | 0.9510 | 0.9479 | 0.9477 | 0.9500 |
| (2) | 0.9421 | 0.9511 | 0.9458 | 0.9302 | 0.9376 | 0.9482 | 0.9533 | 0.9500 | 0.9520 | 0.9517 |
| (3) | 0.9435 | 0.9422 | 0.9455 | 0.9482 | 0.9366 | 0.9472 | 0.9495 | 0.9506 | 0.9511 | 0.9547 |
| (4) | 0.9453 | 0.9484 | 0.9377 | 0.9426 | 0.9328 | 0.9508 | 0.9492 | 0.9497 | 0.9499 | 0.9480 |
| (5) | 0.9470 | 0.9402 | 0.9436 | 0.9448 | 0.9333 | 0.9503 | 0.9496 | 0.9498 | 0.9476 | 0.9478 |
| (6) | 0.9464 | 0.9493 | 0.9412 | 0.9413 | 0.9263 | 0.9474 | 0.9455 | 0.9447 | 0.9483 | 0.9444 |
| (7) | 0.9473 | 0.9496 | 0.9422 | 0.9484 | 0.9325 | 0.9529 | 0.9520 | 0.9537 | 0.9480 | 0.9538 |
| (8) | 0.9482 | 0.9491 | 0.9461 | 0.9375 | 0.9347 | 0.9507 | 0.9520 | 0.9506 | 0.9496 | 0.9495 |
| (9) | 0.9500 | 0.9504 | 0.9489 | 0.9471 | 0.9337 | 0.9509 | 0.9522 | 0.9538 | 0.9543 | 0.9504 |
| (10) | 0.9510 | 0.9500 | 0.9485 | 0.9449 | 0.9370 | 0.9485 | 0.9498 | 0.9507 | 0.9482 | 0.9528 |
| (11) | 0.9498 | 0.9479 | 0.9483 | 0.9402 | 0.9342 | 0.9493 | 0.9554 | 0.9488 | 0.9458 | 0.9454 |
| (12) | 0.9483 | 0.9487 | 0.9482 | 0.9447 | 0.9366 | 0.9511 | 0.9511 | 0.9504 | 0.9502 | 0.9486 |
| (13) | 0.9468 | 0.9473 | 0.9482 | 0.9486 | 0.9408 | 0.9504 | 0.9514 | 0.9466 | 0.9532 | 0.9454 |
| (14) | 0.9405 | 0.9406 | 0.9411 | 0.9432 | 0.9341 | 0.9501 | 0.9498 | 0.9492 | 0.9489 | 0.9501 |
| (15) | 0.9499 | 0.9498 | 0.9492 | 0.9472 | 0.9419 | 0.9506 | 0.9494 | 0.9498 | 0.9496 | 0.9497 |
| (16) | 0.9520 | 0.9499 | 0.9504 | 0.9492 | 0.9440 | 0.9473 | 0.9474 | 0.9469 | 0.9476 | 0.9479 |
| (17) | 0.9494 | 0.9497 | 0.9482 | 0.9502 | 0.9463 | 0.9506 | 0.9498 | 0.9519 | 0.9506 | 0.9504 |
| (18) | 0.9485 | 0.9502 | 0.9476 | 0.9464 | 0.9363 | 0.9478 | 0.9476 | 0.9479 | 0.9497 | 0.9502 |

(1) standard normal; (2) GED(0,1,1); (3) GED(0,1,1.5); (4) GED(0,1,2); (5) SN(-0.68,1.21,1); (6) SN(-1.22,1.58,4); (7) SN(1.22,1.58,-4); (8) SN(0.68,1.21,-1); (9) SN(0,1,0,0); (10) $\text{MVN}_2$(0,0); (11) $\text{MVN}_2$(-0.5,0.5); (12) $\text{MVN}_2$(0.5,0.5); (13) $\text{MVN}_2$(-0.5,-0.5); (14) $\text{MVN}_{10}(\tilde{a}, \tilde{a})$; (15) $\text{MVN}_{10}(\tilde{b}, \tilde{b})$; (16) $\text{MVN}_{10}(\tilde{c}, \tilde{c})$; (17) $\text{MVN}_{10}(\tilde{c}, \tilde{b})$; (18) $\text{MVN}_{10}(\tilde{b}, \tilde{c})$, where $\tilde{a} = (0, ..., 0)$, $\tilde{b} = (0.5, ..., 0.5)$, and $\tilde{c} = (-0.5, 0.5, -0.5, 0.5)$.

Table 3: Coverage probabilities of 95% confidence interval for $Q_\alpha(T)$ based on stochastic volatility sequences. The results are based on 10,000 replicates, and the true $Q_\alpha(T)$ computed by simulating $10^6$ price paths from the true model, $T = 2{,}500$, and $\alpha = 0.99$.

| $\mu$ | known | | | | | unknown | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $U_t \setminus \phi$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
| (1) | 0.9428 | 0.9442 | 0.9450 | 0.9456 | 0.9228 | 0.9504 | 0.9479 | 0.9500 | 0.9495 | 0.9492 |
| (2) | 0.9471 | 0.9446 | 0.9393 | 0.9411 | 0.9344 | 0.9470 | 0.9499 | 0.9508 | 0.9523 | 0.9489 |
| (3) | 0.9512 | 0.9450 | 0.9404 | 0.9460 | 0.9297 | 0.9509 | 0.9519 | 0.9496 | 0.9463 | 0.9535 |
| (4) | 0.9473 | 0.9483 | 0.9481 | 0.9415 | 0.9201 | 0.9492 | 0.9549 | 0.9444 | 0.9470 | 0.9480 |
| (5) | 0.9433 | 0.9443 | 0.9435 | 0.9414 | 0.9273 | 0.9472 | 0.9485 | 0.9504 | 0.9483 | 0.9465 |
| (6) | 0.9191 | 0.9207 | 0.9361 | 0.9218 | 0.9053 | 0.9471 | 0.9434 | 0.9443 | 0.9460 | 0.9449 |
| (7) | 0.9448 | 0.9409 | 0.9381 | 0.9459 | 0.9294 | 0.9554 | 0.9533 | 0.9587 | 0.9573 | 0.9552 |
| (8) | 0.9449 | 0.9447 | 0.9449 | 0.9391 | 0.9268 | 0.9496 | 0.9487 | 0.9481 | 0.9505 | 0.9470 |
| (9) | 0.9414 | 0.9449 | 0.9475 | 0.9424 | 0.9248 | 0.9525 | 0.9497 | 0.9519 | 0.9502 | 0.9467 |
| (10) | 0.9478 | 0.9405 | 0.9473 | 0.9422 | 0.9364 | 0.9491 | 0.9503 | 0.9477 | 0.9502 | 0.9482 |
| (11) | 0.9467 | 0.9448 | 0.9443 | 0.9412 | 0.9247 | 0.9457 | 0.9477 | 0.9534 | 0.9491 | 0.9498 |
| (12) | 0.9453 | 0.9461 | 0.9350 | 0.9392 | 0.9266 | 0.9516 | 0.9505 | 0.9490 | 0.9515 | 0.9481 |
| (13) | 0.9494 | 0.9465 | 0.9468 | 0.9425 | 0.9366 | 0.9509 | 0.9501 | 0.9464 | 0.9496 | 0.9517 |
| (14) | 0.9487 | 0.9472 | 0.9482 | 0.9469 | 0.9436 | 0.9512 | 0.9514 | 0.9516 | 0.9511 | 0.9490 |
| (15) | 0.9468 | 0.9468 | 0.9462 | 0.9443 | 0.9383 | 0.9508 | 0.9523 | 0.9511 | 0.9531 | 0.9516 |
| (16) | 0.9502 | 0.9520 | 0.9501 | 0.9505 | 0.9382 | 0.9475 | 0.9498 | 0.9490 | 0.9503 | 0.9505 |
| (17) | 0.9454 | 0.9449 | 0.9468 | 0.9494 | 0.9443 | 0.9534 | 0.9532 | 0.9524 | 0.9529 | 0.9522 |
| (18) | 0.9482 | 0.9466 | 0.9502 | 0.9481 | 0.9338 | 0.9520 | 0.9524 | 0.9521 | 0.9513 | 0.9493 |

(1) standard normal; (2) GED(0,1,1); (3) GED(0,1,1.5); (4) GED(0,1,2); (5) SN(-0.68,1.21,1); (6) SN(-1.22,1.58,4); (7) SN(1.22,1.58,-4); (8) SN(0.68,1.21,-1); (9) SN(0,1,0,0); (10) $\text{MVN}_2$(0,0); (11) $\text{MVN}_2$(-0.5,0.5); (12) $\text{MVN}_2$(0.5,0.5); (13) $\text{MVN}_2$(-0.5,-0.5); (14) $\text{MVN}_{10}(\tilde{a}, \tilde{a})$; (15) $\text{MVN}_{10}(\tilde{b}, \tilde{b})$; (16) $\text{MVN}_{10}(\tilde{c}, \tilde{c})$; (17) $\text{MVN}_{10}(\tilde{c}, \tilde{b})$; (18) $\text{MVN}_{10}(\tilde{b}, \tilde{c})$, where $\tilde{a} = (0, ..., 0)$, $\tilde{b} = (0.5, ..., 0.5)$, and $\tilde{c} = (-0.5, 0.5, -0.5, 0.5)$.

Table 4: Coverage probabilities of 95% confidence interval for $Q_\alpha(T)$ based on stochastic volatility sequences. The results are based on 10,000 replicates, and the true $Q_\alpha(T)$ computed by simulating $10^6$ price paths from the true model, $T = 5{,}000$, and $\alpha = 0.99$.

| $\mu$ | known | | | | | unknown | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $U_t \setminus \phi$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
| (1) | 0.9495 | 0.9455 | 0.9500 | 0.9484 | 0.9291 | 0.9490 | 0.9507 | 0.9483 | 0.9478 | 0.9502 |
| (2) | 0.9425 | 0.9515 | 0.9463 | 0.9390 | 0.9321 | 0.9480 | 0.9528 | 0.9501 | 0.9522 | 0.9519 |
| (3) | 0.9447 | 0.9447 | 0.9435 | 0.9481 | 0.9378 | 0.9473 | 0.9493 | 0.9502 | 0.9515 | 0.9537 |
| (4) | 0.9435 | 0.9387 | 0.9434 | 0.9411 | 0.9345 | 0.9506 | 0.9488 | 0.9495 | 0.9496 | 0.9477 |
| (5) | 0.9452 | 0.9483 | 0.9458 | 0.9408 | 0.9329 | 0.9500 | 0.9489 | 0.9498 | 0.9473 | 0.9476 |
| (6) | 0.9288 | 0.9495 | 0.9279 | 0.9399 | 0.9260 | 0.9469 | 0.9446 | 0.9437 | 0.9472 | 0.9432 |
| (7) | 0.9472 | 0.9479 | 0.9309 | 0.9421 | 0.9291 | 0.9539 | 0.9533 | 0.9543 | 0.9493 | 0.9543 |
| (8) | 0.9492 | 0.9478 | 0.9465 | 0.9405 | 0.9338 | 0.9511 | 0.9526 | 0.9505 | 0.9496 | 0.9491 |
| (9) | 0.9501 | 0.9503 | 0.9503 | 0.9468 | 0.9352 | 0.9507 | 0.9519 | 0.9538 | 0.9541 | 0.9508 |
| (10) | 0.9503 | 0.9500 | 0.9476 | 0.9460 | 0.9376 | 0.9489 | 0.9503 | 0.9507 | 0.9484 | 0.9524 |
| (11) | 0.9497 | 0.9388 | 0.9466 | 0.9385 | 0.9330 | 0.9491 | 0.9555 | 0.9485 | 0.9464 | 0.9456 |
| (12) | 0.9490 | 0.9477 | 0.9473 | 0.9446 | 0.9343 | 0.9512 | 0.9510 | 0.9503 | 0.9500 | 0.9488 |
| (13) | 0.9455 | 0.9479 | 0.9477 | 0.9402 | 0.9406 | 0.9507 | 0.9509 | 0.9462 | 0.9530 | 0.9453 |
| (14) | 0.9415 | 0.9425 | 0.9451 | 0.9442 | 0.9420 | 0.9496 | 0.9495 | 0.9494 | 0.9490 | 0.9503 |
| (15) | 0.9442 | 0.9459 | 0.9463 | 0.9400 | 0.9339 | 0.9506 | 0.9495 | 0.9500 | 0.9494 | 0.9494 |
| (16) | 0.9476 | 0.9495 | 0.9508 | 0.9485 | 0.9432 | 0.9474 | 0.9474 | 0.9468 | 0.9477 | 0.9479 |
| (17) | 0.9480 | 0.9473 | 0.9470 | 0.9479 | 0.9464 | 0.9507 | 0.9499 | 0.9524 | 0.9505 | 0.9504 |
| (18) | 0.9453 | 0.9495 | 0.9477 | 0.9457 | 0.9360 | 0.9474 | 0.9478 | 0.9477 | 0.9496 | 0.9506 |

(1) standard normal; (2) GED(0,1,1); (3) GED(0,1,1.5); (4) GED(0,1,2); (5) SN(-0.68,1.21,1); (6) SN(-1.22,1.58,4); (7) SN(1.22,1.58,-4); (8) SN(0.68,1.21,-1); (9) SN(0,1,0,0); (10) $\text{MVN}_2$(0,0); (11) $\text{MVN}_2$(-0.5,0.5); (12) $\text{MVN}_2$(0.5,0.5); (13) $\text{MVN}_2$(-0.5,-0.5); (14) $\text{MVN}_{10}(\tilde{a}, \tilde{a})$; (15) $\text{MVN}_{10}(\tilde{b}, \tilde{b})$; (16) $\text{MVN}_{10}(\tilde{c}, \tilde{c})$; (17) $\text{MVN}_{10}(\tilde{c}, \tilde{b})$; (18) $\text{MVN}_{10}(\tilde{b}, \tilde{c})$, where $\tilde{a} = (0, ..., 0)$, $\tilde{b} = (0.5, ..., 0.5)$, and $\tilde{c} = (-0.5, 0.5, -0.5, 0.5)$.

Step 6. Compute $\hat{Q}_\alpha(T)$ based on $H_1$, ..., $H_{1,000}$ from Step 5.

Step 7. Repeat Steps 3 - 6 500 times to get $\hat{Q}_\alpha(T)^1$, ..., $\hat{Q}_\alpha(T)^{500}$.

Step 8. We compute the 95% confidence interval of $Q_\alpha(T)$ based on $\hat{Q}_\alpha(T)^1$, ..., $\hat{Q}_\alpha(T)^{500}$ from Step 7.

Step 9. Check if the confidence interval computed from Step 8 covers the true $Q_\alpha(T)$.

Step 10. Repeat Steps 1 - 9 500 times to get 500 confidence intervals, and see how many confidence intervals cover the true $Q_\alpha(T)$.

For the true data generating processes, we considered the univariate case with $U_t$ being standard normal, GED(0,1,1), and SN(-0.68,1.21,1,0), $T = 500, 1{,}000, 1{,}500, 2{,}000, 2{,}500$, and 5,000, $\mu$ is known and $\mu$ is unknown, $\phi = 0.5$ and 0.9. Again, $\alpha = 0.95$ and 0.99. The other settings are those of Subsection 3.1. The results are summarized in Table 5. In general, the empirical coverage probabilities using (15) and (16) are closer to the nominal coverage probabilities than their simulation-based counterparts. The superiority of using (15) and (16) is pronounced when $\mu$ is unknown. Indeed, for the case that $\mu$ is unknown, all the empirical coverage probabilities of the simulation-based method are lower than 25%. For the case of unknown mean, we attribute the poor performance of the simulation-based method as follows. A close look at the proof of Theorem 2 reveals that the estimation error of $\hat{\mu}_T - \tilde{\mu}$ dominates the limit of $T^{-1/2}\ln(\hat{Q}_\alpha(T)/Q_\alpha(T))$. This error was accumulated to $\exp\{T(\hat{\mu}_T - \tilde{\mu})\}$ in simulating the integrated returns $H = S_0 \exp\{T(\hat{\mu}_T - \tilde{\mu})\}\exp\{\sum_{t=1}^T (r'_t + \mu)\}$ performed in Step 4, where the $r'_t$ are generated by the zero-mean SV model with parameters obtained in Step 2. According to the law of the iterated logarithm, the term $\exp\{T(\hat{\mu}_T - \tilde{\mu})\} = \exp\{\sum_{t=1}^T v_t u_t\}$ fluctuates with probability one between $(\log T)^{-c\sqrt{T}}$ and $(\log T)^{c\sqrt{T}}$ for some positive $c$. Large biases are therefore created in Step 8 by the multiplicative factor $\exp\{T(\hat{\mu}_T - \tilde{\mu})\}$ when computing the lower and upper limits of the confidence interval, and consequently result in low coverage rates of the true quantile.

Table 5: Comparison of the coverage probabilities of 95% confidence intervals for $Q_\alpha(T)$ based on equations (15) and (16) and the traditional sampling-based method, with the parameters estimated by the Bayesian appraoch of Jacquier, Polson and Rossi (1994). The results are based on 500 replicates, and the true $Q_\alpha(T)$ computed by simulating $10^6$ price paths from the true model, $T = 500, 1{,}000, 1{,}500, 2{,}000, 2{,}500$, and 5,000, and $\alpha = 0.95$ and 0.99.

| $\mu$ | | | known | | | | unknown | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | | | 0.95 | | 0.99 | | 0.95 | | 0.99 | |
| $T$ | $\phi$ | $U_t$ | Eqn. (15) | MCMC | Eqn. (15) | MCMC | Eqn. (16) | MCMC | Eqn. (16) | MCMC |
| 500 | 0.5 | (1) | 0.935 | 0.940 | 0.935 | 0.988 | 0.946 | 0.094 | 0.946 | 0.180 |
| | | (2) | 0.916 | 0.842 | 0.916 | 0.902 | 0.946 | 0.098 | 0.944 | 0.174 |
| | | (3) | 0.935 | 0.958 | 0.933 | 0.974 | 0.945 | 0.116 | 0.944 | 0.172 |
| | 0.9 | (1) | 0.891 | 0.880 | 0.889 | 0.940 | 0.946 | 0.108 | 0.945 | 0.190 |
| | | (2) | 0.901 | 0.734 | 0.896 | 0.838 | 0.949 | 0.092 | 0.949 | 0.170 |
| | | (3) | 0.896 | 0.858 | 0.892 | 0.924 | 0.949 | 0.110 | 0.947 | 0.186 |
| 1000 | 0.5 | (1) | 0.939 | 0.996 | 0.939 | 0.998 | 0.950 | 0.110 | 0.950 | 0.198 |
| | | (2) | 0.936 | 0.952 | 0.937 | 0.980 | 0.949 | 0.086 | 0.949 | 0.164 |
| | | (3) | 0.940 | 0.992 | 0.940 | 0.996 | 0.948 | 0.102 | 0.947 | 0.154 |
| | 0.9 | (1) | 0.908 | 0.960 | 0.907 | 0.986 | 0.949 | 0.116 | 0.948 | 0.200 |
| | | (2) | 0.912 | 0.898 | 0.905 | 0.942 | 0.947 | 0.108 | 0.947 | 0.192 |
| | | (3) | 0.913 | 0.962 | 0.912 | 0.994 | 0.948 | 0.094 | 0.948 | 0.182 |
| 1500 | 0.5 | (1) | 0.942 | 0.998 | 0.941 | 1.000 | 0.952 | 0.094 | 0.951 | 0.190 |
| | | (2) | 0.937 | 0.962 | 0.937 | 0.994 | 0.949 | 0.106 | 0.950 | 0.152 |
| | | (3) | 0.945 | 1.000 | 0.943 | 1.000 | 0.950 | 0.092 | 0.950 | 0.158 |
| | 0.9 | (1) | 0.913 | 0.990 | 0.912 | 0.998 | 0.951 | 0.098 | 0.950 | 0.170 |
| | | (2) | 0.917 | 0.942 | 0.918 | 0.984 | 0.951 | 0.112 | 0.950 | 0.182 |
| | | (3) | 0.917 | 0.992 | 0.916 | 0.998 | 0.946 | 0.128 | 0.946 | 0.198 |
| 2000 | 0.5 | (1) | 0.946 | 1.000 | 0.946 | 1.000 | 0.948 | 0.100 | 0.948 | 0.186 |
| | | (2) | 0.939 | 0.976 | 0.935 | 0.996 | 0.950 | 0.134 | 0.950 | 0.212 |
| | | (3) | 0.943 | 1.000 | 0.942 | 1.000 | 0.948 | 0.100 | 0.948 | 0.162 |
| | 0.9 | (1) | 0.927 | 0.994 | 0.926 | 1.000 | 0.949 | 0.122 | 0.948 | 0.202 |
| | | (2) | 0.921 | 0.960 | 0.921 | 0.986 | 0.952 | 0.102 | 0.953 | 0.178 |
| | | (3) | 0.921 | 0.998 | 0.919 | 1.000 | 0.948 | 0.126 | 0.948 | 0.176 |
| 2500 | 0.5 | (1) | 0.945 | 1.000 | 0.945 | 1.000 | 0.950 | 0.094 | 0.950 | 0.168 |
| | | (2) | 0.939 | 0.988 | 0.939 | 0.998 | 0.951 | 0.118 | 0.951 | 0.192 |
| | | (3) | 0.948 | 1.000 | 0.944 | 1.000 | 0.951 | 0.098 | 0.950 | 0.174 |
| | 0.9 | (1) | 0.925 | 1.000 | 0.923 | 1.000 | 0.949 | 0.104 | 0.949 | 0.168 |
| | | (2) | 0.934 | 0.966 | 0.934 | 0.992 | 0.949 | 0.088 | 0.949 | 0.134 |
| | | (3) | 0.926 | 1.000 | 0.927 | 1.000 | 0.947 | 0.122 | 0.947 | 0.194 |
| 5000 | 0.5 | (1) | 0.951 | 1.000 | 0.950 | 1.000 | 0.948 | 0.082 | 0.948 | 0.146 |
| | | (2) | 0.946 | 0.998 | 0.946 | 1.000 | 0.950 | 0.078 | 0.950 | 0.162 |
| | | (3) | 0.944 | 1.000 | 0.946 | 1.000 | 0.950 | 0.096 | 0.950 | 0.186 |
| | 0.9 | (1) | 0.932 | 1.000 | 0.929 | 1.000 | 0.950 | 0.086 | 0.950 | 0.176 |
| | | (2) | 0.938 | 1.000 | 0.932 | 1.000 | 0.952 | 0.106 | 0.952 | 0.178 |
| | | (3) | 0.933 | 1.000 | 0.933 | 1.000 | 0.948 | 0.098 | 0.948 | 0.168 |

(1) standard normal; (2) GED(0,1,1); (3) SN(-0.68,1.2,1,0).

# 4 Application

We applied our proposed method to estimate the integrated return of S&P 500 index and S&P 500 portfolio from CRSP(Center for Research in Security Prices) (available at http://wrds-web.wharton.upenn.edu/wrds/). The data period is from the year of 1963 to the year of 2013, so that we have 12,838 daily data. With the returns $\{r_t\}_{t=1,...,12,838}$, we used the following steps to get the coverage probabilities of 95% confidence intervals for $Q_\alpha(T)$.

Step 1. From $r_1, r_2, ..., r_{1,260}$, we get the first $S_T$.

Step 2. From $r_{21}, r_{22}, ..., r_{1,280}$, we get the second $S_T$.

Step 3. And so on ...

Step 4. Finally, from $r_{11,561}, r_{11,562}, ..., r_{12,820}$, we get the final $S_T$.

Step 5. From Steps 1 to 4, there are 579 $S_T$'s in total.

Step 6. We rank 579 $S_T$'s to get the $\alpha$-th quantile, and treat this quantile computed as the true $Q_\alpha(T)$.

We thus divided the sample into 579 overlapping subsamples with each subsample consisting of 1,260 observations, and every two consecutive subsamples being 20 trading days apart. The 1,260 returns for each subsample is chosen to match the 5-year maturity of the Super LEAPS S&P 500 index option contract mentioned in Section 1. The distance of 20 trading days, about one trading month, that separates neighboring subsamples is only a rough choice intended to strike a balance between generating sufficiently many subsamples and keeping them from too hevily overlapping with each other.

We also used these data to compute the confidence intervals by (16) to see if the confidence intervals cover the true value obtained in Step 8. Finally, we computed the empirical coverage rates. The results are summarized in Table 6. From the table, the empirical coverage rates of the 95% confidence intervals of $Q_\alpha(T)$ are close to the nominal rates. Those for the equal-weighted returns are closer to the nominal rates than those for the value-weighted returns.

Table 6: The true values of $Q_\alpha(T)$ and the empirical coverage rates of the 95% confidence intervals of $Q_\alpha(T)$ for the equal-weighted returns (excluding dividends) and the value-weighted returns (excluding dividends) of the S&P 500 index.

| | EWR | | VWR | |
|---|---|---|---|---|
| $\alpha$ | 0.95 | 0.99 | 0.95 | 0.99 |
| true values of $Q_\alpha(T)$ | 2.5036 | 3.1579 | 2.6936 | 3.1828 |
| empirical coverage rates | 0.9551 | 0.9413 | 0.9119 | 0.9033 |

EWR (equal-weighted returns); VWR (value-weighted returns).

# 5 Discussion

Two issues are worthy of further investigation. First, since VaR is not coherent, it would be interesting from both the theoretical and practical view points to see whether asymptotic properties similar to Theorem 2 can be established for a coherent risk measure such as the conditional tail expectation $E\left(S_T \mid S_T > Q_\alpha(T)\right)$. Second, it is a challenging problem to extend the duration time $T$ from deterministic to random. The extension is motivated by the practice in fund management where fund managers are forced to close parts or all of a fund's positions due to a massive redemption by clients. Since the time at which the redemption occurs is random, the quantile that needs to be evaluated is based on the price process indexed by a random time. The methods developed here may form a good basis to further address the two issues.

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# Supplementary Document

The online Supplement file contains the proofs of Theorem 1 and Theorem 2.

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