EVENT-SEPARABILITY IN THE ELLSBERG URN

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Ellsberg's three-color urn involves two distinct sources of uncertainty – the color composition of the urn (which is subjective) and the identity of the drawn ball (which is objective) – and bets on it can involve mixed objective:subjective uncertainty. It is known that typical betting preferences on this urn violate both the Sure-Thing Principle and probabilistic sophistication over its mixed uncertainty, but are consistent with both of them over its purely subjective uncertainty. In this paper we show that the standard Ellsberg-type preference reversal is actually implied by the Independence Axiom over its purely objective uncertainty.

KEYWORDS: Ellsberg Paradox, ambiguity, ambiguity aversion, event-separability

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1. INTRODUCTION

Separability across mutually exclusive events – the *Independence Axiom* under objective uncertainty and the *Sure-Thing Principle* under subjective uncertainty – has long been considered the hallmark of rational and consistent choice over uncertain prospects, and forms the foundation of the canonical expected utility model of choice under uncertainty.[1] For this reason, empirically or experimentally observed violations of separability – especially systematic violations of this property – are of more concern than they would be in most other branches of consumer theory.[2]

Experimenters have uncovered two systematic and widespread forms of event-nonseparability in decision-makers' preferences. Under objective uncertainty, the Independence Axiom is systematically violated by the *Allais Paradox* and related phenomena. The Sure-Thing Principle is taken to be violated by the subject of this paper, the three-color *Ellsberg Paradox* and similar examples.[3]

In contrast with Allais' purely objective prospects, Ellsberg urns involve both objective and subjective uncertainty, and accordingly allow for three types of uncertain events, partitions and bets: purely subjective (involving the unknown composition of the urn), purely objective (involving the identity of the drawn ball) and mixed objective: subjective (involving the color of the drawn ball). In this paper we show that, rather than being an overarching "metaproperty" of preferences, event-separability turns out to have distinct – and even conflicting – implications when applied to these three different forms of uncertainty.

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[1] Even attempts to model more general 'non-expected utility' preferences have primarily relied upon some notion of event-separability – either applied to a smaller domain of prospects, or else locally in the sense of differentiability.

[2] A possible exception might be intertemporal choice, which relies extensively on separability across time periods.

[3] Allais (1953), Ellsberg (1961). See MacCrimmon and Larsson (1979), Segal (1987), Schmeidler (1989), Starmer (2000), Halevy (2007), Ergin and Gul (2009) and Seo (2009) for surveys, experimental findings, and theoretical responses to these two phenomena.

The following section reviews the Independence Axiom, the Sure-Thing Principle and a third event-separability property known as the *Strong Comparative Likelihood Axiom*. Section 3 illustrates their implications for betting preferences on two simplified versions of the original Ellsberg urn, containing only three balls each. Section 4 extends the argument to the standard 90-ball version, and Section 5 concludes.

2. THREE PRINCIPLES AND A PARADOX

The Independence Axiom, the Sure-Thing Principle and the Strong Comparative Probability Axiom represent three distinct forms of event-separability of preferences over objectively and/or subjectively uncertain prospects.[4] Denoting an objective lottery yielding prize z_i with probability p_i as $\mathbf{P} = (z_1,p_1;\ldots;z_m,p_m)$, the Independence Axiom consists of the property:

(1) $$(z_1^*,p_1^*;\ldots;z_i^*,p_i^*) \succcurlyeq (z_1,p_1;\ldots;z_i,p_i) \quad \Leftrightarrow$$
$$(z_1^*,\alpha p_1^*;\ldots;z_i^*,\alpha p_i^*;\hat{z}_{i+1},(1-\alpha)\hat{p}_{i+1};\ldots;\hat{z}_m,(1-\alpha)\hat{p}_m) \succcurlyeq (z_1,\alpha p_1;\ldots;z_i,\alpha p_i;\hat{z}_{i+1},(1-\alpha)\hat{p}_{i+1};\ldots;\hat{z}_m,(1-\alpha)\hat{p}_m)$$

for all lotteries $(z_1^*,p_1^*;\ldots;z_i^*,p_i^*)$, $(z_1,p_1;\ldots;z_i,p_i)$, $(\hat{z}_{i+1},\hat{p}_{i+1};\ldots;\hat{z}_m,\hat{p}_m)$ and all $\alpha \in (0,1]$. This can be described as separability over objective probability mixtures, in that the individual's ranking of two lotteries is unchanged with they are each mixed with some common third lottery, using some common mixture probability.

The Independence Axiom also implies the following event-separability property:

(1)′ $$(z_1^*,p_1;\ldots;z_i^*,p_i;z_{i+1},p_{i+1};\ldots;z_m,p_m) \succcurlyeq (z_1,p_1;\ldots;z_i,p_i;z_{i+1},p_{i+1};\ldots;z_m,p_m) \quad \Leftrightarrow$$
$$(z_1^*,p_1;\ldots;z_i^*,p_i;\hat{z}_{i+1},p_{i+1};\ldots;\hat{z}_m,p_m) \succcurlyeq (z_1,p_1;\ldots;z_i,p_i;\hat{z}_{i+1},p_{i+1};\ldots;\hat{z}_m,p_m)$$

for all prizes and probabilities. Property (1)′ can be described as separability over objectively uncertain events, in that the individual's ranking of a pair of lotteries is unchanged when one or more of their common prizes $\{z_{i+1},\ldots,z_m\}$ are replaced by the common prizes $\{\hat{z}_{i+1},\ldots,\hat{z}_m\}$, with all other prizes, and all probabilities, unchanged.

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[4] See Savage (1954, p.23, Axiom P2), Samuelson (1952, p.672, Axiom II), and Machina and Schmeidler (1992, p.761, Axiom P4*) for formal statements of these principles.

Denoting a subjective act yielding prize z_j in event E_j as $f(\cdot) = [z_1 \text{ if } E_1;\ldots;z_n \text{ if } E_n]$, the Sure-Thing Principle consists of the property:

$$(2)\quad \begin{array}{l} [z_1^* \text{ if } E_1;\ldots;z_j^* \text{ if } E_j;z_{j+1} \text{ if } E_{j+1};\ldots;z_n \text{ if } E_n] \succcurlyeq [z_1 \text{ if } E_1;\ldots;z_j \text{ if } E_j;z_{j+1} \text{ if } E_{j+1};\ldots;z_n \text{ if } E_n] \;\Leftrightarrow \\ [z_1^* \text{ if } E_1;\ldots;z_j^* \text{ if } E_j;\hat{z}_{j+1} \text{ if } E_{j+1};\ldots;\hat{z}_n \text{ if } E_n] \succcurlyeq [z_1 \text{ if } E_1;\ldots;z_j \text{ if } E_j;\hat{z}_{j+1} \text{ if } E_{j+1};\ldots;\hat{z}_n \text{ if } E_n] \end{array}$$

for all prizes and events. This can be described as separability over subjectively uncertain events, in that the individual's ranking of a pair of acts is unchanged when their common prizes $\{z_{j+1},\ldots,z_n\}$ are replaced by the common prizes $\{\hat{z}_{j+1},\ldots,\hat{z}_n\}$, with all other prizes, and all events, unchanged.

A second, distinct form of separability over subjectively uncertain events is the Strong Comparative Likelihood Axiom, which consists of the property:

$$(3)\quad \begin{array}{l} [z_1^* \text{ if } E_1;z_2^* \text{ if } E_2;z_3 \text{ if } E_3;\ldots;z_n \text{ if } E_n] \succcurlyeq [z_2^* \text{ if } E_1;z_1^* \text{ if } E_2;z_3 \text{ if } E_3;\ldots;z_n \text{ if } E_n] \;\Leftrightarrow \\ [\hat{z}_1^* \text{ if } E_1;\hat{z}_2^* \text{ if } E_2;\hat{z}_3 \text{ if } E_3;\ldots;\hat{z}_n \text{ if } E_n] \succcurlyeq [\hat{z}_2^* \text{ if } E_1;\hat{z}_1^* \text{ if } E_2;\hat{z}_3 \text{ if } E_3;\ldots;\hat{z}_n \text{ if } E_n] \end{array}$$

for all prizes and events for which $z_1^* \succ z_2^*$ and $\hat{z}_1^* \succ \hat{z}_2^*$. This can be described as the existence of well-defined likelihood rankings, in that the individual's preference for which way to stake the more-preferred versus less-preferred of two prizes on some pair of events will not depend upon the two prizes themselves, nor upon the other prizes and events in the act. It has been shown to imply the property of *probabilistic sophistication* – the existence of a subjective probability measure $\mu(\cdot)$ over events such that an act $[z_1 \text{ if } E_1;\ldots;z_n \text{ if } E_n]$ is evaluated solely on the basis of its implied lottery $(z_1,\mu(E_1);\ldots;z_n,\mu(E_n))$.

Although they are both defined over subjective partitions $\{E_1,\ldots,E_n\}$, the Sure-Thing Principle and the Strong Comparative Likelihood Axiom are logically independent properties: the state-dependent expected utility preference function $\Sigma U(z_j|E_j)\mu(E_j)$ (e.g., Karni (1985)) satisfies the former property but not necessarily the latter, whereas the probabilistically sophisticated non-expected utility preference function $V(z_1,\mu(E_1);\ldots;z_n,\mu(E_n))$ (e.g., Machina and Schmeidler (1992)) satisfies the latter property but not necessarily the former.

It is important to note that the above separability properties are taken to hold "for all prizes." Thus, both the Sure-Thing Principle and Strong Comparative Probability Axiom allow the prizes in a subjective act $[\ldots;(\ldots;x_{ij},p_{ij};\ldots)\text{ if }E_j;\ldots]$ to be objective lotteries, and the Independence Axiom allows the prizes in an objective lottery $(\ldots;[\ldots;x_{ij}\text{ if }E_{ij};\ldots],p_i;\ldots)$ to be subjective acts.

It is also important to note that, regardless of what prizes may or may not be assigned to them, the events in a "subjective" partition $\{E_1,\ldots,E_n\}$ may have objective attributes. Thus, for the three-color urn of Table 1, the event red will be purely objective with probability 1/3, and the events black and yellow will have the attribute that their union has an objective probability of 2/3. We term a partition $\{E_1,\ldots,E_n\}$ *purely subjective* if it contains no objective information whatsoever.

TABLE 1 THREE-COLOR ELLSBERG PARADOX

	30 balls	60 balls		
	red	black	yellow	
$f_1(\cdot)$	\$100	\$0	\$0	$\succ$
$f_2(\cdot)$	\$0	\$100	\$0	
$f_3(\cdot)$	\$100	\$0	\$100	$\prec$
$f_4(\cdot)$	\$0	\$100	\$100	

As noted above, the Sure-Thing Principle is violated by a choice problem known as the *Three-Color Ellsberg Paradox* and by similar examples due to or inspired by Ellsberg (1961). As illustrated in Table 1, this example consists of an urn containing 90 balls, 30 of which are known to be red, with each of the remaining 60 either black or yellow, although the number of black versus yellow balls is unknown. A ball is drawn from this urn, and the bet $f_1(\cdot)$ = [\$100 if red; \$0 if black; \$0 if yellow] yields \$100 if it is red, \$0 if it is black and \$0 if it is yellow, similarly for the bets $f_2(\cdot)$, $f_3(\cdot)$ and $f_4(\cdot)$. When faced with these prospects, most individuals express a strict preference for $f_1(\cdot)$ over $f_2(\cdot)$ and for $f_4(\cdot)$ over $f_3(\cdot)$, as indicated to the right of the table. But since bets $f_3(\cdot)$ and $f_4(\cdot)$ can be obtained from $f_1(\cdot)$ and $f_2(\cdot)$ by replacing their common outcome [\$0 if yellow] by the common outcome [\$100 if yellow], these preferences violate the Sure-Thing Principle over the mixed objective:subjective partition $\{\text{red}, \text{black}, \text{yellow}\}$. Since $f_1(\cdot) \succ f_2(\cdot)$ would reveal the likelihood ranking $\mu(\text{red}) > \mu(\text{black})$ but $f_3(\cdot) \prec f_4(\cdot)$ would reveal $\mu(\text{red}) < \mu(\text{black})$, this pair of rankings also violates the Strong Comparative Likelihood Axiom, and hence probabilistic sophistication, over the partition $\{\text{red}, \text{black}, \text{yellow}\}$.[5]

The intuition behind these preferences is clear: $f_1(\cdot)$ offers the \$100 prize on an objective 1/3-probability event {red}, whereas $f_2(\cdot)$ offers it on one element of an informationally symmetric but subjective partition {black, yellow} of a 2/3-probability event. Similarly, $f_4(\cdot)$ offers the prize on an objective 2/3-probability event, whereas $f_3(\cdot)$ offers it on the union of a 1/3-probability event and the other element of that subjective partition. Such preferences are termed *ambiguity averse*.

3. ANALYSIS OF TWO SIMPLIFIED EXAMPLES

For simplicity, we start with a simpler version of the three-color urn, involving only three balls and including two additional bets, as illustrated in Table 2. For reasons similar to those in the standard ninety-ball version, we define *Ellsberg preferences* (or an *Ellsberg reversal*) as the rankings $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ coupled with $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$. Thus, in addition to the standard rankings, we posit indifference between the informationally-symmetric acts $f_2(\cdot)$ and $f_2'(\cdot)$, and between the informationally-symmetric acts $f_3(\cdot)$ and $f_3'(\cdot)$.

[5] If we require that subjective beliefs satisfy $\mu(\text{red}) = 1/3$, then the rankings $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ themselves would violate probabilistic sophistication over the mixed partition $\{\text{red}, \text{black}, \text{yellow}\}$, since they would imply $1/3 = \mu(\text{red}) > \mu(\text{black}) = \mu(\text{yellow})$ even though $\mu(\text{red}) + \mu(\text{black}) + \mu(\text{yellow}) = 1$.

TABLE 2 THREE-COLOR ELLSBERG PARADOX (SIMPLIFIED VERSION)

	1 ball	2 balls		
	red	black	yellow	
$f_1(\cdot)$	\$100	\$0	\$0	$\succ$
$f_2(\cdot)$	\$0	\$100	\$0	$\sim$
$f_2'(\cdot)$	\$0	\$0	\$100	
$f_3'(\cdot)$	\$100	\$100	\$0	$\sim$
$f_3(\cdot)$	\$100	\$0	\$100	$\prec$
$f_4(\cdot)$	\$0	\$100	\$100	

Bets on an Ellsberg urn are most clearly represented when its mixed objective:subjective partition {red, black, yellow} is replaced by an "orthogonal" representation of the urn's underlying objective and subjective uncertainty. The orthogonal representation of mixed prospects – which dates back to the equations of Anscombe and Aumann (1963) and the diagrams of Pratt, Raiffa and Schlaifer (1964) – has been applied to Ellsberg urns by Segal (1987), Klibanoff, Marinacci and Mukerji (2005), Nau (2006), Ergin and Gul (2009) and Seo (2009), and we adopt this approach here.

To separately represent the underlying objective and subjective uncertainty in this urn, we follow Ergin and Gul (2009) and label objective events by assigning serial numbers to the individual balls, and subjective events by the composition (number of black versus yellow balls) of the urn. Ball number 1 is red, and balls 2 and 3 can take the respective colors black:black, black:yellow, yellow:black or yellow:yellow. The objective uncertainty thus consists of the serial number i of the drawn ball, whereas the subjective uncertainty consists of which of the states BB, BY, YB or YY obtains. For example, the event that ball number 2 is actually drawn is purely objective, with probability 1/3, and the event that (drawn or not) its color is black is purely subjective, depending on whether or not the one of the states BB or BY obtains. The event that a black ball – any black ball – is drawn is thus mixed objective:subjective, depending upon whether or not the serial number and state are one of the pairs (2,BB), (3,BB), (2,BY) or (3,YB) .

TABLE 3 ORTHOGONAL REPRESENTATION OF THE THREE-COLOR PARTITION

serial number of drawn ball		BB	BY	YB	YY
	3	black	yellow	black	yellow
	2	black	black	yellow	yellow
	1	red	red	red	red
		composition of the urn			

Table 3 displays this orthogonal representation of the urn's objective, subjective and mixed uncertainty, with the serial number of the chosen ball represented vertically, and the black:yellow composition of the urn represented horizontally. This twelve-element orthogonal partition is seen to be a refinement of the three-element mixed partition {red,black,yellow} of Table 2. Table 4 depicts bets $f_1(\cdot)$, $f_2(\cdot)$, $f_2'(\cdot)$, $f_3'(\cdot)$, $f_3(\cdot)$ and $f_4(\cdot)$ as bets over this orthogonal partition. Reading square-by-square, each bet can be viewed as a bet over the joint objective and subjective realizations, with sure consequences as prizes. Reading column-by-column, each can be viewed as a subjective act over the space {BB,BY,YB,YY}, with objective lotteries as prizes. Reading row-by-row, each can be viewed as an objective lottery, with subjective acts over {BB,BY,YB,YY} as prizes.[6]

[6] That such tables allow for these three perspectives was noted by Pratt, Raiffa and Schlaifer (1964,p.364).

TABLE 4 ORTHOGONAL REPRESENTATION OF THE THREE-COLOR BETS

f_1: [\$100 if R; \$0 if B; \$0 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$0	\$0	\$0	\$0
2	\$0	\$0	\$0	\$0
1	\$100	\$100	\$100	\$100

$\succ$

f_2: [\$0 if R; \$100 if B; \$0 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$100	\$0	\$100	\$0
2	\$100	\$100	\$0	\$0
1	\$0	\$0	\$0	\$0

$\sim$

f_2': [\$0 if R; \$0 if B; \$100 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$0	\$100	\$0	\$100
2	\$0	\$0	\$100	\$100
1	\$0	\$0	\$0	\$0

f_4: [\$0 if R; \$100 if B; \$100 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$100	\$100	\$100	\$100
2	\$100	\$100	\$100	\$100
1	\$0	\$0	\$0	\$0

$\succ$

f_3': [\$100 if R; \$100 if B; \$0 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$100	\$0	\$100	\$0
2	\$100	\$100	\$0	\$0
1	\$100	\$100	\$100	\$100

$\sim$

f_3: [\$100 if R; \$0 if B; \$100 if Y]

serial number of drawn ball	BB	BY	YB	YY
3	\$0	\$100	\$0	\$100
2	\$0	\$0	\$100	\$100
1	\$100	\$100	\$100	\$100

composition of the urn

Table 4 illustrates how event-separability over these different forms of uncertainty has different implications for how preferences for $f_1(\cdot)$ versus $f_2(\cdot)$ are linked to preferences for $f_3(\cdot)$ versus $f_4(\cdot)$. By providing an alternative illustration of how these acts are defined over the mixed partition {red, black, yellow}, it provides an alternative illustration of the above argument that the Ellsberg rankings of $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ coupled with $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$ violate both the Sure-Thing Principle and the Strong Comparative Probability Axiom (and thus probabilistic sophistication) over the urn's *mixed objective:subjective uncertainty.*

Table 4 also illustrates why the Ellsberg rankings of $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ coupled with $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$ *do not* violate event-separability – neither the Sure-Thing Principle nor the Strong Comparative Probability Axiom – over the urn's *purely subjective* *uncertainty* (its unknown composition). The reason is that when applied to its purely subjective uncertainty, these two forms of event-separability *have no implications* for preferences across the two triples of bets. In contrast to the mixed partition {red, black, yellow}, the urn's purely subjective partition {BB,BY,YB,YY} corresponds to the four columns of each matrix. In order for the ranking $f_1(\cdot) \succ f_2(\cdot)$ or the ranking $f_1(\cdot) \succ f_2'(\cdot)$ to imply anything about the ranking of $f_4(\cdot)$ versus $f_3(\cdot)$ or of $f_4(\cdot)$ versus $f_3'(\cdot)$ under the Sure-Thing Principle, one of these latter pairs of bets would have to differ from one of the former pairs solely by the replacement of some common prize or prizes (objective lotteries) by some other common prize or prizes. As seen from the table, this is not the case.[7] In order for the Strong Comparative Probability Axiom to have any implication, $f_1(\cdot)$ would have to differ from $f_2(\cdot)$ or from $f_2'(\cdot)$ by swapping the prizes (objective lotteries) over two of the events BB, BY, YB or YY, which is also not the case. The compatibility of the Sure-Thing Principle and the Strong Comparative Probability Axiom over the urn's subjective uncertainty with Ellsberg preferences has been known since the work of Klibanoff, Marinacci and Mukerji (2005) and Nau (2006), who have shown how Ellsberg-type ambiguity aversion is consistent with a subjectively-separable preference function of the form $\sum_{j=1}^{n} g(\sum_{i=1}^{m} U(x_{ij}) \cdot p_{ij}) \cdot \mu(E_j)$.

That Ellsberg preferences violate event-separability over mixed partitions but not purely subjective partitions should come as no surprise, since the latter would be a strictly weaker property. What does come as a surprise, however, is that given the typical Ellsberg rankings of $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ over the first triple of acts, the reversed rankings of $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$ over the second triple are actually *implied* by event-separability (that is, by the Independence Axiom) over the urn's *purely objective uncertainty.*

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[7] The only prize shared by bets $f_1(\cdot)$ and $f_2(\cdot)$ (or by $f_1(\cdot)$ and $f_2'(\cdot)$) over the purely subjective partition {BB,BY,YB,YY} is the objective lottery (\$100,⅓;\$0,⅔) in the events BY and YB, which is indeed replaced by the objective lottery (\$100,⅔;\$0,⅓) in bets $f_3(\cdot)$, $f_3'(\cdot)$ and $f_4(\cdot)$. However, for the Sure-Thing Principle to make a prediction, one of the pairs $(f_3(\cdot),f_4(\cdot))$ or $(f_3'(\cdot),f_4(\cdot))$ would have to differ from one of the pairs $(f_1(\cdot),f_2(\cdot))$ or $(f_1(\cdot),f_2'(\cdot))$ *only* by this replacement, which is not the case.

As seen in Table 4, bets $f_4(\cdot)$ and $f_3'(\cdot)$ respectively differ from $f_1(\cdot)$ and $f_2(\cdot)$ by the replacement of a common 1/3 probability of the act [\$0 if BB; \$0 if BY; \$0 if YB; \$0 if YY] with a 1/3 probability of the act [\$100 if BB; \$100 if BY; \$100 if YB; \$100 if YY], with all remaining prizes (acts) and their probabilities unchanged. By the Independence Axiom (separability over purely objective events), preferences must satisfy the relationship $f_1(\cdot) \succcurlyeq (\succ)\, f_2(\cdot) \Leftrightarrow f_4(\cdot) \succcurlyeq (\succ)\, f_3'(\cdot)$, and therefore also the relationship $f_1(\cdot) \succcurlyeq (\succ)\, f_2(\cdot) \Leftrightarrow f_4(\cdot) \succcurlyeq (\succ)\, f_3(\cdot)$. Anyone who satisfies the Independence Axiom over the urn's purely objective uncertainty and who strictly prefers $f_1(\cdot)$ over $f_2(\cdot)$ *must* exhibit the Ellsberg reversal and strictly prefer $f_4(\cdot)$ over $f_3(\cdot)$.[8]

The urn of Table 2 is offered as a 1:30 "scale replica" of the original 90-ball urn to allow for the orthogonal representation of Table 4, and as such it retains the information structure of the original urn, namely the individual color uncertainty of each non-red ball. In correspondence, Robert Nau has observed that by adopting a slightly different information structure, we can construct an urn with an even simpler orthogonal representation, which exhibits the same event-separability properties as the urn of Table 2 and Ellsberg's 90-ball urn.

Modify the urn of Table 2 so that the second and third balls are either both black or both yellow, but keep the description of the six bets the same, so that the Ellsberg rankings of $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ coupled with $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$ continue to violate the Sure-Thing Principle over the mixed objective-subjective partition {red, black, yellow}. Table 5 presents an orthogonal representation of the six bets, which is essentially Table 4

[8] Why not avoid the need for Table 4, and make this argument by simply going back to Table 2, expressing its bets as the objective lotteries $f_1(\cdot)$ = (\$100,⅓; \$100,⅔), $f_2(\cdot)$ = (\$0,⅓; [\$100 if black; \$0 if yellow],⅔), $f_3'(\cdot)$ = (\$100,⅓; [\$100 if black; \$0 if yellow],⅔), $f_4(\cdot)$ = (\$0,⅓; \$100,⅔), and applying the Independence Axiom directly to those ⅓ chances of \$0 or \$100? The reason is that the "prize" [\$100 if black; \$0 if yellow] is not a well-defined subjective act, since the partition {black, yellow} is not exhaustive. To be well-defined, we must write $f_2(\cdot)$ = (\$0,⅓; [\$100 if BB; \$0 if BY; \$100 if YB; \$0 if YY],⅓; [\$100 if BB; \$100 if BY; \$0 if YB; \$0 if YY],⅓), and similarly for $f_3'(\cdot)$.

with the states BY and YB omitted. Since $f_1(\cdot)$ shares no common column with either $f_2(\cdot)$ or $f_2'(\cdot)$, nor does it differ from them by the swapping of any columns, the Ellsberg rankings do not violate either the Sure-Thing Principle or the Strong Comparative Probability Axiom over the urn's purely subjective uncertainty. But since $f_4(\cdot)$ and $f_3'(\cdot)$ can be respectively obtained from $f_1(\cdot)$ and $f_2(\cdot)$ by replacing a 1/3 probability of the act [\$0 if BB; \$0 if YY] with a 1/3 probability of the act [\$100 if BB; \$100 if YY], the Independence Axiom continues to imply $f_1(\cdot) \succcurlyeq (\succ)\, f_2(\cdot) \Leftrightarrow f_4(\cdot) \succcurlyeq (\succ)\, f_3'(\cdot)$ and therefore $f_1(\cdot) \succcurlyeq (\succ)\, f_2(\cdot) \Leftrightarrow f_4(\cdot) \succcurlyeq (\succ)\, f_3(\cdot)$. Once again, anyone who strictly prefers $f_1(\cdot)$ over $f_2(\cdot)$ and who satisfies the Independence Axiom over purely objective uncertainty must strictly prefer $f_4(\cdot)$ over $f_3(\cdot)$. Although this example does not retain the ball-by-ball uncertainty of the original Ellsberg urn, it exhibits the same event-separability properties.

TABLE 5 ORTHOGONAL REPRESENTATION OF THE THREE-COLOR BETS (NAU'S EXAMPLE)

f_1: [\$100 if R; \$0 if B; \$0 if Y] $\succ$ f_2: [\$0 if R; \$100 if B; \$0 if Y] $\sim$ f_2': [\$0 if R; \$0 if B; \$100 if Y]

serial number of drawn ball	f_1: BB	f_1: YY		f_2: BB	f_2: YY		f_2': BB	f_2': YY
3	\$0	\$0		\$100	\$0		\$0	\$100
2	\$0	\$0	$\succ$	\$100	\$0	$\sim$	\$0	\$100
1	\$100	\$100		\$0	\$0		\$0	\$0

f_4: [\$0 if R; \$100 if B; \$100 if Y] $\succ$ f_3': [\$100 if R; \$100 if B; \$0 if Y] $\sim$ f_3: [\$100 if R; \$0 if B; \$100 if Y]

serial number of drawn ball	f_4: BB	f_4: YY		f_3': BB	f_3': YY		f_3: BB	f_3: YY
3	\$100	\$100		\$100	\$0		\$0	\$100
2	\$100	\$100	$\succ$	\$100	\$0	$\sim$	\$0	\$100
1	\$0	\$0		\$100	\$100		\$100	\$100

(BB, YY: composition of the urn)

4. ANALYSIS OF THE 90-BALL URN

The arguments of the above section can be extended to the 90-ball urn of Table 1. As before, we include the bets $f_2'(\cdot)$ = [\$0 if red; \$0 if black; \$100 if yellow] and $f_3'(\cdot)$ = [\$100 if red; \$100 if black; \$0 if yellow] and define Ellsberg preferences as the rankings $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ coupled with $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$. From the table, it is clear that these preferences violate both the Sure-Thing Principle and the Strong Comparative Probability Axiom over the urn's mixed objective:subjective uncertainty (the partition {red, black, yellow}), and as the above-cited authors have shown, they are consistent with these two forms of event-separability over the urn's purely subjective uncertainty (its unknown composition). Here we generalize the argument of the previous section to demonstrate that the Ellsberg reversal is again implied by the Independence Axiom over the urn's purely objective uncertainty, namely the identity of the drawn ball.

Label the balls so that balls 1,…,30 are red and each of balls 31,…,90 is either yellow or black, and define …*BBYB*… as the state that balls 31 through 90 take the respective colors …,*B*,*B*,*Y*,*B*,… etc., so that there are 2^{60} states. Define the event $\mathcal{B}_i$ as the union of all states in which ball i is black, and event $\mathcal{Y}_i$ as its complement, that is, the set of all states in which ball i is yellow. The orthogonalized partition of the full objective: subjective uncertainty of the 90-ball urn thus consists of all pairs of the form $(i, \ldots BBYB \ldots)$. The color of the drawn ball will be red if ball $i \in \{1,\ldots,30\}$ is drawn, it will be black if ball $i \in \{31,\ldots,90\}$ is drawn and the state is in $\mathcal{B}_i$, and it will be yellow if ball $i \in \{31,\ldots,90\}$ is drawn and the state is in $\mathcal{Y}_i$.

Each bet of the form $f(\cdot)$ = [\$R if red; \$B if black; \$Y if yellow] can thus be represented as an objective lottery with subjective acts as prizes:

(4) $$(\$R, {}^1\!/\!{}_3; \ldots; [\$B \text{ if } \mathcal{B}_i; \$Y \text{ if } \mathcal{Y}_i], {}^1\!/\!{}_{90}; \ldots)_{i=31,\ldots,90}$$

so that the bets of Table 1, along with $f_2'(\cdot)$ and $f_3'(\cdot)$, can be represented as:

$$
(5)\quad
\begin{aligned}
f_1(\cdot) &= (\$100,\tfrac{1}{3};\ldots;[\$0 \text{ if } \mathcal{B}_i;\$0 \text{ if } \mathcal{Y}_i],\tfrac{1}{90};\ldots) = (\$100,\tfrac{1}{3};\$0,\tfrac{2}{3})\\
f_2(\cdot) &= (\$0,\tfrac{1}{3};\ldots;[\$100 \text{ if } \mathcal{B}_i;\$0 \text{ if } \mathcal{Y}_i],\tfrac{1}{90};\ldots)\\
f_2'(\cdot) &= (\$0,\tfrac{1}{3};\ldots;[\$0 \text{ if } \mathcal{B}_i;\$100 \text{ if } \mathcal{Y}_i],\tfrac{1}{90};\ldots)\\
f_3'(\cdot) &= (\$100,\tfrac{1}{3};\ldots;[\$100 \text{ if } \mathcal{B}_i;\$0 \text{ if } \mathcal{Y}_i],\tfrac{1}{90};\ldots)\\
f_3(\cdot) &= (\$100,\tfrac{1}{3};\ldots;[\$0 \text{ if } \mathcal{B}_i;\$100 \text{ if } \mathcal{Y}_i],\tfrac{1}{90};\ldots)\\
f_4(\cdot) &= (\$0,\tfrac{1}{3};\ldots;[\$100 \text{ if } \mathcal{B}_i;\$100 \text{ if } \mathcal{Y}_i],\tfrac{1}{90};\ldots) = (\$100,\tfrac{2}{3};\$0,\tfrac{1}{3})
\end{aligned}
$$

As in the three-ball urn, bet $f_3'(\cdot)$ is obtained from $f_2(\cdot)$ by replacing a 1/3 probability of the prize \$0 with a 1/3 probability of the prize \$100, with all remaining prizes (subjective acts) and their probabilities unchanged. Bet $f_4(\cdot) = (\$100,⅔;\$0,⅓)$ is obtained from $f_1(\cdot) = (\$100,⅓;\$0,⅔)$ by precisely this same replacement. By the Independence Axiom, preferences must again satisfy the relationship $f_1(\cdot) \succcurlyeq (\succ)\, f_2(\cdot) \Leftrightarrow f_4(\cdot) \succcurlyeq (\succ)\, f_3'(\cdot)$ and therefore the relationship $f_1(\cdot) \succcurlyeq (\succ)\, f_2(\cdot) \Leftrightarrow f_4(\cdot) \succcurlyeq (\succ)\, f_3(\cdot)$. Anyone who satisfies the Independence Axiom over purely objective uncertainty and who strictly prefers $f_1(\cdot)$ over $f_2(\cdot)$ in the 90-ball urn must again exhibit the Ellsberg reversal and strictly prefer $f_4(\cdot)$ over $f_3(\cdot)$.

5. CONCLUSION

The event-separability properties of the Ellsberg preferences of $f_1(\cdot) \succ f_2(\cdot) \sim f_2'(\cdot)$ coupled with $f_3'(\cdot) \sim f_3(\cdot) \prec f_4(\cdot)$ in a three-color Ellsberg urn can be summarized as follows:

- Such preferences *do* violate both the Sure-Thing Principle and the Strong Comparative Probability Axiom (and therefore probabilistic sophistication) over the urn's *mixed objective:subjective uncertainty* – namely, the color of the drawn ball.
- Such preferences *do not* violate either Sure-Thing Principle or Strong Comparative Probability over its *purely subjective uncertainty* – namely, its unknown composition.

- Such preferences are in effect *implied* by the Independence Axiom over its *purely objective uncertainty* – namely of the identity of the drawn ball.[9]

As mentioned, the first two of these points are well known – the purpose of this paper is to point out the third, and thus the need for care when positing event-separability under objective, subjective and/or mixed uncertainty.

[9] We say "in effect" since the Independence Axiom would also allow for the opposite, *ambiguity loving* reversal of $f_1(\cdot) \prec f_2(\cdot) \sim f_2'(\cdot)$ coupled with $f_3'(\cdot) \sim f_3(\cdot) \succ f_4(\cdot)$. Either way, the implications of the Independence Axiom over the urn's purely objective uncertainty are either distinct from, or in contrast to, the implications of the Sure-Thing Principle over its purely subjective or its mixed objective:subjective uncertainty.

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