Moderate Interactions in Games with Induced Complementarities*

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Abstract

We consider a finite population simultaneous move game with heterogeneous interaction modes across different pairs of players. We allow for general interaction patterns, but restrict our analysis to games whose pure strategy Nash equilibrium conditions boil down to a set of piece-wise linear conditions, so that an equilibrium is a solution to a linear complementarity problem.

We introduce a new class of games for which a suitable linear transformation of the original interaction matrix induces a game with complementarities. We provide general moderation conditions on the interaction matrix such that a game in this class has a unique Nash equilibrium, that we are able to characterize by means of a closed-form expression involving a generalized version of the Katz network measure of node centrality.

Keywords: Nash equilibrium, uniqueness, complementarity, moderation, interaction matrix, linear complementarity problem. **JEL Classification:** A14, C72, L14.

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1 Introduction

It is a feature of most economic groupings that the behavior of each member may affect the behavior and well-being of every other member. This cross influence can sometimes be exerted through some public good nature of the group interaction. In a given competitive industry, for instance, the common market price at which all firms sell their output, and that enters their individual profit calculations, results from the production decision of each such firm. It can also be directly embodied in the preferences of the agents. When cross influences operate, every single action taken by an individual in isolation can affect the well-being of other individuals in the group. Cross influences thus naturally yield to interdependent decisions.

The aim of this paper is to analyze the equilibrium behavior for general interaction patterns, where cross influences are allowed to vary in sign and value across different pairs of players.

More precisely, we consider a finite population simultaneous move game with heterogeneous interaction modes across different pairs of players. When an interaction is positive (negative), the decisions of the linked agents are said to be strategic complements (substitutes). Beyond its differences in sign, interactions can also differ in intensity across different pairs of players. For a given population of players, we gather together the characteristics of each possible bilateral interaction (sign and intensity) in a matrix, the interaction matrix. We consider very general interaction matrices, that can reflect either strategic complementarity or strategic substitutability in actions of any intensity, or both, for a same group of players and depending on the pair of players considered. Although we allow for general interaction patterns, we restrict our analysis to situations where best replies are continuous piece-wise linear functions. For these games, the interaction matrix essentially reflects the second-order derivatives of individual payoffs. With such payoff structure, best-response functions are piece-wise linear. Linear oligopoly models, and bayesian games or team problems with linear information structures, for instance, fall into this category.[1]

Borrowing from the extensive literature on *complementarity problems*, we find conditions on the interaction matrix such that our game with heterogeneous interaction modes has a unique Nash equilibrium in pure strategies. The uniqueness and existence of a solution is a desirable property in order to be able to make unambiguous predictions and to make consistent comparative statics. The most general property that guarantees existence of a unique Nash equilibrium is the *moderation* of the interaction matrix of the game, that is,

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[1] See, e.g., Vives (1999), Radner (1962) and Calvó-Armengol and De Martí (2007) for an application.

the fact that substitutability effects (even from the own actions of players) balance the complementarities in the interactions.

In addition to providing a simple characterization of the pure-strategy Nash equilibrium, we also give a closed-form expression for this equilibrium that involves a generalized version of the Katz network measure of node centrality. This somehow generalizes some of the results in Ballester *et al.* (2006). By expressing the Nash equilibrium of a game as a function of the Katz-centrality in a network we build a bridge between the game theoretical and the sociological literatures.

In this paper, we pay attention to a specific interaction environment whose structure allows for a simple characterization of existence and uniqueness of equilibrium. More precisely, we introduce a new class of games where complementarities, although not apparent, may be *induced* through a particular transformation applied to the interaction matrix of the game.

This class of games includes in particular games with *complementarities*, for which the cross-payoffs derivatives between every pair of players are non-negative. We identify these games with what we call *signed* interaction matrices. Games with complementarities have been extensively dealt with in the literature. Supermodular games, for instance, correspond to games with complementarities with a compact lattice strategy space.[2] In the class of games we consider, we deal with an un-bounded strategy space, the non-negative real line. Without boundedness, we lose some properties which are characteristic of supermodular games, such as the lattice structure of the Nash equilibrium set (Zhou, 1994). But some other interesting properties, like existence and uniqueness of the equilibrium correspondence, are obtained from simple and intuitive conditions on the pattern of interactions, related to the spectral radius of the interaction matrix.

However interesting, games with complementarities constitute only a subclass of the richer family of games that we consider. This family includes all those games whose interaction matrix can be mapped into a signed matrix by a linear transformation. This broader and new class of games includes, for instance, the public good network game analyzed by Bramoullé and Kranton (2007), where players' actions are strategic substitutes across network-linked players. The games with both local complementarities and global substitutabilities analyzed in Ballester *et al.* (2006) constitute another example of a game where complementarities can be induced by means of a simple linear transformation on the original interaction matrix.

We also investigate games with *substitutabilities*, where incentives to increase one's action are lower, the lower the effort of the other players. It turns out that, in order to check for

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[2] In our case, this is equivalent to a bounded strategy space. See Topkis (1979), Vives (1990) and Milgrom and Roberts (1990).

existence and uniqueness of an equilibrium we can employ the very same spectral procedure proposed for games with complementarities. For instance, in the local (network) public good game analyzed in Bramoullé and Kranton (2007), a Nash equilibrium exists and is unique when the network of relationships is not very dense, i.e., if its spectral radius is low enough.

Finally, we analyze a subclass of games with induced complementarities that we call games with shifted complementarities. Games with shifted complementarities are obtained from the original interaction matrix by operating a signed upwards shift. In this case, the pattern of induced complementarities emerges by wiping out the substitutabilities that are latent in the original pattern of cross influences through a simple translation upwards of the entries of the interaction matrix. In other words, games with shifted complementarities can be additively decomposed into local (network) complementarities and substitutabilities. For this class of games, the equilibrium isomorphism between the original game and the induced game is particularly simple: equilibrium actions are proportional to each other in the two games. The additive shift in the interaction matrix thus translates into a multiplicative shift in the equilibrium actions. Besides, this multiplicative isomorphism almost characterizes the whole class of games with shifted complementarities. Here, again, the Katz-centrality characterizes equilibrium behavior. These results generalize previous findings in Ballester *et al.* (2006).

Although the linear form of the complementarity problem (or the linear-quadratic form of the utilities) is a considerably strong assumption, we want to stress that many results of existence and uniqueness in the linear complementarity theory have been used to derive existence and uniqueness in the non-linear case. For instance, in Kolstad and Mathiesen (1987) and Simsek *et al.* (2005) the uniqueness of a solution is essentially determined by the property of uniqueness in the linear complementarity problem induced by the Jacobian of the function at every point.[3] Further boundedness conditions allow these authors to prove the existence of an equilibrium. Thus, our analysis may constitute the starting point for a more general analysis of games with arbitrary (but smooth enough) utility functions.

The paper is organized as follows. Section 2 introduces the model and defines the interaction patterns. Section 3 deals with the class of games for which moderation of the interactions turns out to characterize existence and uniqueness of pure-strategy Nash equilibrium. In Section 4, we establish a direct link between the equilibrium and the Katz-centrality of a network. Section 5 is devoted to some special subclasses of games: games with complementarities, games with substitutabilities and games with shifted complementarities. Section 6

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[3] See, also, Mas-Colell (1979). Notable exceptions to this approach are Bamon and Frayssé (1985) and Rosen (1965).

concludes.

2 The model

Each player i in the set $N = \{1, \ldots, n\}$ has to simultaneously decide on her un-bounded and non-negative level of effort $x_i \in \mathbb{R}_+$. Any vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ denotes a profile of strategies for the players. We focus on twice-differentiable utility functions

$$u_i : \mathbb{R}_+^n \to \mathbb{R},\ i = 1, \ldots, n$$

such that each u_i is concave in own effort: $\partial^2 u_i (\mathbf{x}) / \partial x_i^2 \leq 0$ for all $\mathbf{x} \in \mathbb{R}_+^n$.

We focus on those games whose pure strategy Nash equilibria $\mathbf{x}^*$ are characterized by the linear complementarity problem:

$$\begin{aligned} \mathbf{x} &\geq \mathbf{0} \\ \mathbf{u}^0 - \mathbf{\Gamma x} &\leq \mathbf{0} \\ \mathbf{x}^\intercal \left(\mathbf{u}^0 - \mathbf{\Gamma x} \right) &= 0 \text{ for all } i, \end{aligned} \tag{1}$$

where the row vector that corresponds to $\mathbf{x}$ is written as the transpose $\mathbf{x}^\intercal$.

The first inequality in (1) states that strategies are in the adequate strategy space, $\mathbb{R}_+^n$. The second inequality essentially checks for non-positive marginal utilities at equilibrium. The third equality (combined with the two previous ones) states that a negative marginal utility for player i implies a corner action $x_i = 0$ for this player at equilibrium.

Notice, in particular, that $\mathbf{x} = \mathbf{0}$ solves the linear complementarity problem when $\mathbf{u}_0 \leq \mathbf{0}$.

Within this framework, a game can be referred to as a pair $(\mathbf{u}^0, \mathbf{\Gamma})$, where $\mathbf{u}^0$ is a vector of $\mathbb{R}^n$ and $\mathbf{\Gamma} = \left[\gamma_{ij} \right]$ is the $n \times n$ *interaction matrix* of the game, an element of $\mathbb{R}^n \times \mathbb{R}^n$.

The coordinates of the vector $\mathbf{u}^0$ summarize individual incentives when every other player exerts a zero effort. If $u_i^0 \geq 0$ ($u_i^0 \leq 0$), then player i wants to increase (decrease) his action when all other players play zero.

The interaction matrix $\mathbf{\Gamma}$ describes how agents' incentives interrelate with each other: $-\gamma_{ij}$ characterizes the direction and the intensity of the externality exerted by player j's action on player i's payoffs. When $-\gamma_{ij} \geq 0$ ($-\gamma_{ij} \leq 0$), we say that player j's action is a strategic complement (substitute) for player i, i.e., player j exerts a positive (negative) externality on player i.

We give two examples of economic problems that fit into this set-up.

Example **1** *Suppose that individual payoffs are linear-quadratic and given by:*

$$u_i(\mathbf{x}) = \alpha_i + \frac{1}{2}\sigma_{ii}x_i^2 + \sum_{j\neq i}\sigma_{ij}x_ix_j, \textit{ for all } i \in N, \tag{2}$$

and are concave in own-action, that is, $\sigma_{ii} \leq 0$ *for all* i*. It is straightforward to see that all the Nash equilibria of this game are given by the solutions to (1) where* $u_i^0 = \alpha_i$ *and* $\gamma_{ij} = -\sigma_{ij}$*, for all* i *and* j*. This game is analyzed in Ballester et at. (2006). Linear oligopoly models fit into this payoff class (Vives, 1999). For instance, a simple model of Cournot competition with heterogeneous goods, linear inverse demand and quadratic cost functions of the form*

$$\begin{aligned} P_i(\mathbf{x}) &= a_i - \sum_{j=1}^{n}\eta_{ij}x_j \\ C_i(\mathbf{x}) &= c_ix_i^2, \end{aligned}$$

where x_i *is the amount of good produced by firm* i*, is a special case of (2).*

Example **2** *Consider a finite population of players connected by a network of relationships* $\mathbf{G} = (g_{ij})$*, where* $g_{ij} \geq 0$ *measures the strength of the social link between* i *and* j*, and* $g_{ii} = 0$*. Each player* i *contributes* $x_i \in \mathbb{R}_+$ *units of effort to the provision of a public good, and utility is given by:*

$$u_i(\mathbf{x}; \mathbf{G}) = b(x_i + \sum_{j\neq i} g_{ij}x_j) - cx_i, \tag{3}$$

where $c > 0$ *is the marginal cost of contribution,* $b' > 0$ *and* $b'' < 0$*. It is assumed that there exists a contribution level* $\widetilde{x} \in \mathbb{R}_+$ *such that* $b'(\widetilde{x}) = c$*. This local public good game is analyzed in Bramoullé and Kranton (2007). It can be shown that all the Nash equilibria of this game are given by the solutions to (1) where* $u_i^0 = \widetilde{x}$*,* $\gamma_{ii} = 1$ *and* $\gamma_{ij} = g_{ij}$*, for all* i *and* $j \neq i$*.*

3 Existence and uniqueness

We say that the interaction matrix $\mathbf{\Gamma}$ *yields a unique equilibrium* when the games $(\mathbf{u}^0, \mathbf{\Gamma})$ admit a unique Nash equilibrium in pure strategies for all vectors $\mathbf{u}^0$.

We make the following assumption that we maintain throughout:

(A1) $\gamma_{ii} \geq 0$, for all $i \in N$.

Condition (A1) amounts to requiring concavity in own-action (see, e.g., Example 1). We now give two useful definitions.

Definition **1** *We say that an interaction matrix* $\mathbf{\Gamma}$ *is signed if all of its out-of-diagonal terms are non-positive, that is,* $\gamma_{ij} \leq 0$ *for all* $i \neq j$*,*

Signed interaction matrices correspond to situations where actions are strategic complements, i.e., where $(\mathbf{u}^0, \mathbf{\Gamma})$ is a *game with complementarities*. Intuitively, when a matrix is signed, players' incentives are aligned upwards. Observe that any signed matrix $\mathbf{\Gamma}$ can be written as

$$\mathbf{\Gamma} = s\mathbf{I} - \mathbf{G},$$

where s is a scalar and $\mathbf{G} \geq \mathbf{0}$ is a non-negative matrix.

For any two vectors $\mathbf{x}$ and $\mathbf{y}$, we write $\mathbf{x} >> \mathbf{y}$ to denote that $x_i > y_i$ for all i.

Definition **2** *We say that an interaction matrix* $\mathbf{\Gamma}$ *is moderate if* $\mathbf{\Gamma z} >> \mathbf{0}$ *for some non-negative vector* $\mathbf{z} \geq \mathbf{0}$*.*

In games with moderate interactions, the externality exerted to each player by the rest of the population (γ_{ij}, $j \neq i$) and by his own effort (γ_{ii}) balance each other so that the game can be regarded as having *no* complementarities on average. Indeed, moderation requires that there exists a collection of non-negative weights such that all the weighted sums of the row terms in the interaction matrix are non-negative, when the same weights are used for every sum.

Moderation thus ensures that the strength of complementarities is modest in the game, for instance, by balancing off-diagonal versus diagonal values in the interaction matrix. There are a number of *ad hoc* conditions that do so. A standard condition in the economics literature is diagonal dominance which, in its simplest version, requires that:

$$\gamma_{ii} > \sum_{j \neq i} \left|\gamma_{ij}\right|, \text{ for all } i = 1, ..., n. \tag{4}$$

Moderation is a weaker requirement than diagonal dominance.[4]

The following result characterizes existence and uniqueness of pure strategy Nash equilibria of $(\mathbf{u}^0, \mathbf{\Gamma})$, for all $\mathbf{u}^0$.

Theorem **1** *Let* $\mathbf{T}$ *be a signed matrix such that the matrix* $\mathbf{\Gamma T}$ *is signed and moderate. Then,* $\mathbf{\Gamma}$ *yields a unique equilibrium if and only if* $\mathbf{\Gamma}$ *is moderate.*

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[4] Indeed, any matrix satisfying (A1) with a dominant diagonal is moderate because $\mathbf{M1} >> \mathbf{0}$.

Consider first the case where $(\mathbf{u}^0, \mathbf{\Gamma})$ is a game with complementarities, i.e., the interaction matrix $\mathbf{\Gamma}$ is signed. By taking $\mathbf{T}$ equal to the identity matrix, Theorem 1 states that existence and uniqueness is equivalent to requiring that $\mathbf{\Gamma}$ be also moderate, that is, that own-concavity γ_{ii} compensates for (moderates) the complementarities in the actions of the rest of players. Theorem 1 applies to more general games than just games with complementarities: if we can map linearly $\mathbf{\Gamma}$ through a signed transformation into a signed and moderate matrix, existence and uniqueness of a Nash equilibrium reduces to the interaction matrix $\mathbf{\Gamma}$ being moderate.

Any game with substitutabilities, like in (3), trivially has a moderate interaction matrix because $\mathbf{\Gamma} \geq \mathbf{0}$. In this case, the key to uniqueness and existence is to find a signed transformation $\mathbf{T}$ that induces a signed and moderate game $\mathbf{\Gamma T}$. We will reexamine this type of games in Section 5.2.

The conclusions of Theorem 1 also hold when $\mathbf{\Gamma T}$ is signed and not moderate as long as $\mathbf{T}$ is both signed and moderate.[5]

Finally, it should be mentioned that existence and uniqueness are here independent of the nature (sign and value) of the incentives that operate at the origin, i.e., of the particular choice of the vector $\mathbf{u}^0$. In Theorem 2, we concentrate on the subclass of games with $\mathbf{u}^0 \geq \mathbf{0}$ and provide additional characteristics of the equilibrium.

To conclude, Theorem 1 builds a link between the concepts of existence and uniqueness of pure-strategy Nash equilibrium. To see this, note that $\mathbf{\Gamma}$ is moderate if and only if the game $(\mathbf{u}^0, \mathbf{\Gamma})$ admits at least one Nash equilibrium for some $\mathbf{u}^0 >> \mathbf{0}$. Then, the result in Theorem 1 implies that existence and uniqueness are equivalent notions whenever incentives at the origin are strictly positive. This can be stated formally: either the game $(\mathbf{u}^0, \mathbf{\Gamma})$ has a unique equilibrium *for all* $\mathbf{u}^0$, or the game $(\mathbf{u}^0, \mathbf{\Gamma})$ has *no* equilibrium *for all* $\mathbf{u}^0 >> \mathbf{0}$.

4 Characterization

4.1 Katz centrality

Consider some non-negative matrix $\mathbf{G} \geq \mathbf{0}$ such that $0 \leq g_{ij} \leq 1$, for all $i, j \in N$. We can interpret $\mathbf{G}$ as the adjacency matrix of a weighted and directed network on N, where the direct link ij receives the weight g_{ij}. If $g_{ij} = g_{ji}$, for all $i \neq j$, we say that the network is undirected. If $g_{ij} \in \{0, 1\}$, for all $i, j \in N$, we say that the network is un-weighted. If $g_{ii} = 0$,

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[5] More generally, in Pang (1979) it is proven that the equivalence holds whenever the following general condition holds: there exist vectors $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n_+$ such that $\mathbf{r}^t\mathbf{T} + \mathbf{s}^t\mathbf{\Gamma T} >> \mathbf{0}$.

the network has no self-loops.

Notice that un-weighted and undirected networks without self-loops are only one particular example of the more general (weighted, directed and with self-loops) networks.

Let $\mathbf{u} \geq \mathbf{0}$ be a non-negative vector.

Definition **3** *The vector of* $\mathbf{u}$*-Katz centralities of parameter* $a \geq 0$ *in the network* $\mathbf{G}$ *is:*

$$\mathbf{b_u}\left(a\mathbf{G}\right) = \left[\mathbf{I} - a\mathbf{G}\right]^{-1}\mathbf{u} = \sum_{p=0}^{+\infty} a^p \mathbf{G}^p \mathbf{u}. \tag{5}$$

Here, it is assumed that a is small enough, so that the inverse exists and can be written as the infinite series in (5). In particular, $a\rho\left(\mathbf{G}\right) < 1$, where $\rho\left(\cdot\right)$ denotes the spectral radius[6]. A central result in matrix algebra shows that, under this condition, the inverse $\left[\mathbf{I} - a\mathbf{G}\right]^{-1}$ is nonnegative, so that the $\mathbf{u}$-Katz centralities are nonnegative.

Based on this spectral condition, a geometric interpretation of the Katz centralities is the following. The matrix $\mathbf{G}$ keeps track of (possibly weighted) network direct links. More generally, the matrix $\mathbf{G}^p$ keeps track of all the paths of length p in this network (possibly with an intensity reflecting the weight of the links on any such path). Let first $\mathbf{u} = \mathbf{1}$, and suppose $\mathbf{b_1}\left(a\mathbf{G}\right) \geq \mathbf{0}$. Then, the Katz centrality of player i counts all paths of any length stemming from i in the network, weighted by the geometrically decaying factor a. For general $\mathbf{u}$, paths reaching arbitrary node j are pondered by u_j.

More generally, consider an arbitrary non-negative matrix $\mathbf{G} \geq \mathbf{0}$. We can easily rescale it in order to obtain a network $\mathbf{G}'$. Let $\lambda_{\mathbf{G}} = \max\{g_{ij} : i, j = 1, ..., n\}$. We can then write $\mathbf{G} = \lambda_{\mathbf{G}}\mathbf{G}'$ with $0 \leq g'_{ij} \leq 1$, for all $i, j \in N$, where $\mathbf{G}'$ is the adjacency matrix of a weighted and directed network on N. Abusing slightly, we write $\mathbf{b_u}\left(\mathbf{G}\right) = \mathbf{b_u}\left(\lambda_{\mathbf{G}}\mathbf{G}'\right)$.

4.2 Nash is Katz

Given an arbitrary matrix $\mathbf{G}$, its spectral radius $\rho\left(\mathbf{G}\right)$ is given by the largest modulus of its eigenvalues. When $\mathbf{G} \geq \mathbf{0}$, the spectral radius is a standard measure of the density of the network $\mathbf{G}$. As a matter of fact, it is an increasing function of the intensities of the links of the network.

A signed interaction matrix $\mathbf{\Gamma}$ can always be decomposed as $\mathbf{\Gamma} = s\mathbf{I} - \mathbf{G}$, where $\mathbf{G} \geq \mathbf{0}$ is a non-negative matrix, and s is a scalar. This decomposition is not uniquely defined, of course.

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[6] Given an arbitrary matrix $\mathbf{G}$, its spectral radius $\rho\left(\mathbf{G}\right)$ is given by the largest modulus of its eigenvalues.

Lemma 1 *Let* $\mathbf{\Gamma} = s\mathbf{I} - \mathbf{G}$ *signed. Then,* $\mathbf{\Gamma}$ *is moderate if and only if* $s > \rho(\mathbf{G})$.

This spectral radius characterization of a moderate matrix is a well-known result in linear algebra and it is independent of its decomposition.[7] Recall that a matrix that satisfies (A1) and is diagonally dominant is always moderate. The spectral radius characterization of moderation for a signed matrix is a weaker requirement than diagonal dominance.

We are now ready to characterize the pure strategy Nash equilibrium actions. As before section, we deal with a signed transformation $\mathbf{T}$ applied to the interaction matrix $\mathbf{\Gamma}$.

Theorem 2 *Suppose that* $\mathbf{\Gamma}$ *is a moderate interaction matrix and that the interaction matrix* $\mathbf{\Gamma T}$ *is signed and moderate, i.e., that we can write* $\mathbf{\Gamma T} = s\mathbf{I} - \mathbf{G}$, *with* $\mathbf{G} \geq \mathbf{0}$ *and* $s > \rho(\mathbf{G})$. *Then, if* $\mathbf{u}_0 \geq \mathbf{0}$ *and* $\mathbf{T}\mathbf{b}_{\mathbf{u}_0}(\mathbf{G}/s) \geq \mathbf{0}$, *the unique pure strategy Nash equilibrium* $\mathbf{x}^*$ *of the game* $(\mathbf{u}_0, \mathbf{\Gamma})$ *is:*

$$\mathbf{x}^* = \frac{1}{s}\mathbf{T}\mathbf{b}_{\mathbf{u}_0}(\mathbf{G}/s).$$

A number of comments are in order.

First, the equilibrium of the game is essentially a transformation of the Katz centrality index of the network $\mathbf{G}/s$, which captures the structure of interrelations in the transformed interaction matrix $\mathbf{\Gamma T}$. In Section 5.3, we explore more in details the resulting interaction matrix $\mathbf{\Gamma T}$, which is directly related to a game with complementarities. In this specific class of games, the action exerted by each player is essentially given by its centrality in the network $\mathbf{G}/s$, i.e., the more intense the (positive) externalities exerted upon a player, the higher his effort level at equilibrium.

Second, consider the matrix $s\mathbf{I} - \mathbf{G}$. We know from Lemma 1 that this matrix is moderate when a spectral radius inequality holds. It turns out that this very same spectral radius inequality is a necessary and sufficient condition for the inverse matrix $[\mathbf{I} - s\mathbf{G}]^{-1}$ to be non-negative (Debreu and Herstein, 1953). Therefore, under the spectral radius condition, the $\mathbf{u}_0$-Katz centrality (5) vector is non-negative, $\mathbf{b}_{\mathbf{u}_0}(\mathbf{G}/s) \geq \mathbf{0}$, and the geometric intuition where equilibrium actions are related to the length of network paths holds.

Third, given a signed matrix $\mathbf{T}$ and a non-negative vector $\mathbf{y}$, it is not always granted that $\mathbf{Ty}$ is non-negative. This observation, together with the fact that the strategy space is the non-negative orthant $\mathbb{R}^n_+$, justifies the extra sign condition $\mathbf{T}\mathbf{b}_{\mathbf{u}_0}(s\mathbf{G}) \geq \mathbf{0}$ in Theorem 2.

Finally, letting $\mathbf{T}$ be equal to the identity matrix, Theorem 2 implies that the pure strategy Nash equilibrium of the game with interaction matrix $s\mathbf{I} - \mathbf{G}$ coincides with the Katz centrality vector $(1/s)\,\mathbf{b}_{\mathbf{u}_0}(\mathbf{G}/s)$. Therefore, for general $\mathbf{T}$, Theorem 2 states that the

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[7] That is, let $\mathbf{\Gamma} = (1/s)\,\mathbf{I} - \mathbf{G} = (1/s')\,\mathbf{I} - \mathbf{G}'$. Then, $1 > s\rho(\mathbf{G})$ is equivalent to $1 > s'\rho(\mathbf{G}')$.

pure strategy Nash equilibrium of the original game with interaction matrix $\mathbf{\Gamma}$ is *isomorphic* to the equilibrium of an induced game with interaction matrix $\mathbf{\Gamma T}$. The one-to-one mapping between those two equilibria is linear, and involves the very same transformation matrix $\mathbf{T}$ that maps the interaction matrices of the original and the induced game with each other. Notice, however, that this linear correspondence only holds when $\mathbf{Tb}_{\mathbf{u}_0}(s\mathbf{G}) \geq \mathbf{0}$. That is, even when $(\mathbf{u}_0, \mathbf{\Gamma})$ has a unique Nash equilibrium, there need not be a simple linear correspondence between this equilibrium and that of the induced game with interaction matrix $\mathbf{\Gamma T}$.

5 Three families of games

5.1 Games with complementarities

Recall that $(\mathbf{u}_0, \mathbf{\Gamma})$ is said to have complementarities whenever its interaction matrix $\mathbf{\Gamma}$ is signed. For instance, consider the game with linear-quadratic payoffs in Example 1. The interaction matrix $\mathbf{\Gamma}$ is given by the cross derivatives:

$$\gamma_{ij} = -\frac{\partial^2 u_i}{\partial x_i \partial x_j},$$

A signed interaction matrix is equivalent to having $\gamma_{ij} \leq 0$ for all $i \neq j$, that is, non-negative cross effects.

Given $S \subset N$ and a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$, let $\mathbf{M}_S$ be the restriction of $\mathbf{M}$ to the rows and columns in S. For an arbitrary vector $\mathbf{y} \in \mathbb{R}^n$, let $P(\mathbf{y}) = \{i \in N : y_i > 0\}$ collect the indices of the rows of $\mathbf{y}$ with a positive coordinate. Similarly, for each $\mathbf{y} \in \mathbb{R}^n$, let $\mathbf{y}^+ \in \mathbb{R}_{++}^{|P(\mathbf{y})|}$ be the vector where all nonpositive entries of $\mathbf{y}$ have been eliminated. For instance, if $\mathbf{y} = (5, 4, -3) \in \mathbb{R}^3$, then $P(\mathbf{y}) = \{1, 2\}$ and $\mathbf{y}^+ = (5, 4)$.

Theorems 1 and 2 boil down to the following result.

Corollary **1** *Consider a signed interaction matrix* $\mathbf{\Gamma} = s\mathbf{I} - \mathbf{G}$, *with* $\mathbf{G} \geq \mathbf{0}$. *Then:*

- $\mathbf{\Gamma}$ *yields a unique equilibrium if and only if* $s > \rho(\mathbf{G})$,
- *In this case, the unique pure strategy Nash equilibrium* $\mathbf{x}^*$ *of the game* $(\mathbf{u}_0, \mathbf{\Gamma})$ *is:*

$$\begin{aligned} x_i^* &= \frac{1}{s} b_{i,\mathbf{u}_0^+}\left(\mathbf{G}_{P(\mathbf{u}_0)}/s\right), \textit{ for all } i \in P(\mathbf{u}_0) \\ x_i^* &= 0 \textit{ otherwise.} \end{aligned}$$

Thus, in the case with complementarities, existence and uniqueness of A Nash equilibrium is directly determined by the density of the interactions in the matrix $\mathbf{\Gamma} = s\mathbf{I} - \mathbf{G}$, that is, by the spectral radius of $\mathbf{G}/s$. The intuition is the following. With non-negative cross effects, upward shifts in players' actions feed positively into each other. If these cross effects are moderate, these feedback loops dampen, and players' actions eventually reach some equilibrium point. But, if these cross effects are too big, the positive feed-back loops can trigger an un-bounded escalation in individual actions, and equilibrium fails to exist.[8]. Precisely, the spectral radius condition $s > \rho(\mathbf{G})$ bounds these cross effects in a way that accounts both for their size and for their pattern.

Indeed, when $\mathbf{G}$ is the adjacency matrix of an un-weighted and un-directed network without self-loops, its spectral radius is also a measure of the irregularity of the network geometry. The higher $\rho(\mathbf{G})$, the more irregular the network, the more concentrate the complementarities in a handful of players, and the higher the required own-concavity term $1/s$ for equilibrium to exist. The following example illustrates this point.

Example **3** *Consider minimally connected networks, also referred to as trees. The most irregular tree is the star. The most regular tree is the line. Both networks, though, have the same total number of links, $n-1$. In the star, the central node reaps complementarities from many different sources (all the peripheral players), while in the line the playing field is more even. The spectral radius for the star and the line are, respectively, $\sqrt{n-1}$ and $2\cos\frac{\pi}{n+1}$. In accordance with intuition, the equilibrium existence and uniqueness condition is more demanding for the star than for the line, that is, $\sqrt{n-1} > 2\cos\frac{\pi}{n+1}$.*

For games with complementarities, the Katz closed-form expression for equilibrium actions allows for clear-cut comparative statics where the monotonicity of actions is tied down to the pattern of complementarities across players.

More precisely, consider two games with complementaries with a unique equilibrium, $(\mathbf{u}_0, \mathbf{\Gamma}_1)$ and $(\mathbf{u}_0, \mathbf{\Gamma}_2 = \mathbf{\Gamma}_1 - \mathbf{B})$, with $\mathbf{B} \geq \mathbf{0}$. Denote Nash equilibrium actions as $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$, respectively. Complementarities are stronger in $\mathbf{\Gamma}_2$ than in $\mathbf{\Gamma}_1$ across every pair of players. Corollary 1, together with the fact that Katz centralities increase with the network density implies that $\mathbf{x}_2^* \geq \mathbf{x}_1^*$. In words, individual equilibrium actions increase when pair-wise complementarities are strengthened.

The monotonicity in $\mathbf{u}_0$ can be established similarly.

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[8] Unless the strategy space is arbitrarily bounded from above, of course, in which case we can borrow directly from the literature on supermodular games.

5.2 Games with substitutabilities

We say that $(\mathbf{u}_0, \mathbf{\Gamma})$ is a game with strategic substitutabilities when $\gamma_{ij} \geq 0$. For instance, consider the network public good game in Example 2. The interaction matrix of this game $\mathbf{\Gamma} = \mathbf{I} + \mathbf{G}$ corresponds to a game with strategic substitutabilities.

A game with strategic substitutabilities has non-negative out-of-diagonal terms. It is thus trivially moderate. Following Theorem 1, the key to existence and uniqueness of a pure strategy Nash equilibrium for a game with strategic substitutabilities is to find a signed linear transformation that induces a signed and moderate game.

We describe a natural candidate for this signed linear transformation.

The interaction matrix of a game with substitutabilities can be decomposed as $\mathbf{\Gamma} = s\mathbf{I}+\mathbf{G}$, with $\mathbf{G} \geq \mathbf{0}$ without any loss of generality.

Let $\mathbf{T} = s\mathbf{I} - \mathbf{G}$. This is a signed matrix obtained from the original interaction matrix by (roughly) changing the sign of its off-diagonal terms. The interaction matrix $\Gamma\mathbf{T} = s^2\mathbf{I} - \mathbf{G}^2$ is obviously signed, and corresponds to a game with complementarities. When players behave as followers that best-respond to others' best-responses, local network substitutabilities induce complementarities for players who are two-links-away from each other in the network.

Theorems 1 and 2 amount to the following result.

Proposition 1 *Consider an interaction matrix with Substitutabilities* $\mathbf{\Gamma} = s\mathbf{I} + \mathbf{G}$, *with* $\mathbf{G} \geq \mathbf{0}$. *Then:*

- $\mathbf{\Gamma}$ *yields a unique equilibrium if* $s > \rho(\mathbf{G})$,
- *In this case, if* $\mathbf{u}_0 \geq \mathbf{0}$ *and* $(s\mathbf{I} - \mathbf{G})\, \mathbf{b}_{\mathbf{u}_0}(\mathbf{G}^2/s^2) \geq \mathbf{0}$, *then the unique pure strategy Nash equilibrium* $\mathbf{x}^*$ *of the game* $(\mathbf{u}_0, \mathbf{\Gamma})$ *is:*

$$\mathbf{x}^* = \frac{1}{s^2}(s\mathbf{I} - \mathbf{G})\, \mathbf{b}_{\mathbf{u}_0}\left(\mathbf{G}^2/s^2\right).$$

The following examples confront the condition for existence and uniqueness to the condition for the Nash-Katz characterization to hold.

Example 4 *Consider the game in Example 2, and consider a regular network* $\mathbf{G}$ *such that* $\sum_{j=1}^n g_{ij} = d$, *for all* $i = 1, ..., n$. *Then,* $\rho(\mathbf{G}) = d$, *and the existence and uniqueness condition becomes* $d < 1$. *Straight algebra gives* $\mathbf{b}_{\alpha\mathbf{1}}(\mathbf{G}^2) = \alpha\mathbf{1}/(1 - d^2)$. *The Katz-Nash characterization holds trivially, and all players contribute* $x_i^* = \alpha/(1 + d)$ *at the unique and interior equilibrium.*

Example **5** *Consider again the game in Example 2, and let* $\mathbf{G}$ *be the star centered on player 1, with* $n \geq 2$*:*

$$\mathbf{G}=\begin{bmatrix} 0 & \delta & \cdots & \delta \\ \delta & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \delta & 0 & \cdots & 0 \end{bmatrix} \text{ and } \mathbf{G}^2=\begin{bmatrix} \delta^2(n-1) & 0 & \cdots & 0 \\ 0 & \delta^2 & \cdots & \delta^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \delta^2 & \cdots & \delta^2 \end{bmatrix},$$

where $\delta > 0$ *is an intensity factor. The existence and uniqueness condition is* $\delta\sqrt{n-1} < 1$*. Straight algebra gives* $\mathbf{b}_{\alpha\mathbf{1}}(\mathbf{G}^2) = \alpha\mathbf{1}/\left(1-\delta^2(n-1)\right)$*. We can then conclude the following. When* $1/(n-1) \leq \delta < 1/\sqrt{n-1}$*, the unique pure strategy Nash equilibrium is partially corner, and given by* $x_1^* = 0$ *and* $x_i^* = \alpha$*, for all* $i \geq 2$*.*[9] *That is, for a high value of the substitutability term,* δ*, the hub of the star free-rides on the spokes who contribute the optimal effort for the individual players. When* $\delta \leq 1/(n-1)$*, the unique pure strategy Nash equilibrium is interior and given by* $x_1^* = \alpha(1-\delta(n-1))/\left(1-\delta^2(n-1)\right)$ *and* $x_i^* = \alpha(1-\delta)/\left(1-\delta^2(n-1)\right)$*, for all* $i \geq 2$*. That is, for a low value of the substitutability term,* δ*, all (hub and spoke) players contribute at the unique equilibrium.*

5.3 Games with shifted complementarities

We say that a game has *shifted complementarities* when a suitable downwards translation of its interaction matrix induces a game with complementarities. The substitutability shift that decreases the value of the out-of-diagonal terms of the interaction matrix is required to be of rank one.

Formally, $\mathbf{\Gamma}$ is has shifted complementarities if there exist $\mathbf{u}, \mathbf{v} \geq \mathbf{0}$, such that $\mathbf{\Gamma} - \mathbf{u}\mathbf{v}^\intercal$ is signed.

The following result summarizes the equilibrium properties of games with shifted complementarities, when induced complementarities are moderate.

Proposition **2** *Consider an interaction matrix* $\mathbf{\Gamma}$ *such that* $\mathbf{\Gamma} - \mathbf{u}\mathbf{v}^\intercal$ *is signed and moderate for some* $\mathbf{u}, \mathbf{v} \geq \mathbf{0}$*. Then,* $\mathbf{\Gamma}$ *yields a unique equilibrium. Moreover, the unique equilibrium of the game* $(\mathbf{u}, \mathbf{\Gamma})$ *is interior and given by:*

$$\mathbf{x}^* = \frac{1}{1+\mathbf{v}^\intercal\mathbf{y}^*}\mathbf{y}^*, \tag{6}$$

where $\mathbf{y}^*$ *is the Katz-Nash equilibrium of the game with complementarities* $(\mathbf{u}, \mathbf{\Gamma} - \mathbf{u}\mathbf{v}^\intercal)$*.*

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[9] See, also, Proposition 2 in Bramoullé and Kranton (2007) for this equilibrium.

Reciprocally, suppose that the game $(\mathbf{u}, \mathbf{\Gamma})$ *admits some equilibrium* $\mathbf{x}^*$*, for some* $\mathbf{u} >> \mathbf{0}$*, and that* $\mathbf{\Gamma} - \mathbf{u}\mathbf{v}^\intercal$ *is signed for some* $\mathbf{v} \geq \mathbf{0}$ *such that* $\mathbf{v}^\intercal \mathbf{x}^* < 1$*. Then,* $\mathbf{\Gamma} - \mathbf{u}\mathbf{v}^\intercal$ *yields a unique equilibrium. In particular, the unique equilibrium* $\mathbf{y}^*$ *of the game* $(\mathbf{u}, \mathbf{\Gamma} - \mathbf{u}\mathbf{v}^\intercal)$ *satisfies (6).*

For games with shifted complementarities, the equilibrium isomorphism between the original game and the associated shifted (complementarities) game takes a particularly simple form. We can write the shifted complementarity interaction matrix as $\mathbf{\Gamma} - \mathbf{u}\mathbf{v}^\intercal = s\mathbf{I} - \mathbf{G}$, with $\mathbf{G} \geq \mathbf{0}$. The equilibrium action in the original game is *proportional* to the Katz-Nash equilibrium of the induced (shifted) game given in Corollary 1:

$$\begin{aligned} y_i^* &= \frac{1}{s} b_{i,\mathbf{u}^+}\left(\mathbf{G}_{P(\mathbf{u})}/s\right), \text{ if } u_i > 0 \\ y_i^* &= 0 \text{ if } u_i = 0. \end{aligned}$$

The proportionality factor is then

$$\frac{1}{1 + \mathbf{v}^\intercal \mathbf{y}^*},$$

identical for all players. In other words, *the additive shift in substitutabilities leads to a multiplicative shift in the equilibrium actions*. In particular, the relative equilibrium actions across players is the same for the original and for the associated shifted game.

The reason is the following. The rank-one shift wipes out a common substitutability term from the original game. The resulting induced game with complementarities has an interaction matrix $s\mathbf{I} - \mathbf{G}$ that determines the values of the relative equilibrium actions. The impact of the rank-one global substitutability term has then a level effect on these actions, common to all players.

This is well-illustrated in the example below.

EXAMPLE **6** *Consider the game* $(\mathbf{1}, \mathbf{\Gamma})$ *with interaction matrix:*

$$\mathbf{\Gamma} = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & 1 \\ 1 & 1 & 3/2 \end{bmatrix}.$$

Notice that this matrix does not have a dominant diagonal. We operate an $\mathbf{1}\mathbf{1}^\intercal$ *rank-one shift that yields the following interaction matrix corresponding to a game with complementarities:*

$$\mathbf{\Gamma} - \mathbf{1}\mathbf{1}^\intercal = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

This is a signed and moderate matrix (in fact, it is diagonal dominant). The equilibrium $\mathbf{y}^*$ *for the induced game* $(\mathbf{1}, \mathbf{\Gamma} - \mathbf{1}\mathbf{1}^\intercal)$ *is* $\mathbf{y}^{*\intercal} = (2,2,1)/2$. *The equilibrium* $\mathbf{x}^*$ *for the original game* $(\mathbf{1}, \mathbf{\Gamma})$ *is proportional to* $\mathbf{y}^*$, *with a proportionality factor* $1/(1+\mathbf{y}^{*\intercal}\mathbf{1}) = 2/7$. *Therefore,* $\mathbf{x}^{*\intercal} = (2/7)\,\mathbf{y}^{*\intercal} = (2,2,1)/7$.

Proposition 2 establishes uniqueness and interiority for games with shifted complementarities. A sufficient condition is that one can find a rank one shift that produces a signed and moderate matrix. As a matter of fact, this condition almost completely characterizes all games with shifted complementarities that have a unique and interior equilibrium, as stated in the second part of Proposition 2. The next example shows the role of the condition $\mathbf{v}^\intercal \mathbf{x}^* < 1$.

Example **7** *Consider the interaction matrix:*

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

This interaction matrix yields a unique equilibrium. In particular, when $\mathbf{u}_0 >> \mathbf{0}$, *the unique equilibrium of* $(\mathbf{u}_0, \mathbf{\Gamma})$ *is interior and given by:*

$$\mathbf{x}^* = (u_{01} + u_{03}, u_{01} + u_{02}, u_{02} + u_{03})/2.$$

The "best" candidate vector $\mathbf{v}$ *to operate a rank-one* $\mathbf{u}_0\mathbf{v}^\intercal$ *is* $\mathbf{v}^\intercal = (1/u_{03}, 1/u_{01}, 1/u_{02})$, *where* $\mathbf{v}^\intercal \mathbf{x}^* > 3/2 > 1$. *Therefore, the unique Nash equilibrium of the game* $(\mathbf{u}_0, \mathbf{\Gamma})$ *is not proportional to Katz centrality. As a matter of fact, a sequence of rank-one shifts leaves at best with the following signed matrix:*

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

which is not moderate.

The key to Proposition 2 is to operate the "right" rank one shift $\mathbf{u}_0\mathbf{v}^\intercal$ so that the resulting matrix $\mathbf{\Gamma} - \mathbf{u}\mathbf{v}^\intercal$ is both signed and moderate.

The signed condition amounts to having $\gamma_{ij} \leq u_i v_j$ for all $i \neq j$, which asks for (roughly) vectors $\mathbf{u}, \mathbf{v}$ with high enough coordinate values. The second condition requires that the complementarities in $\mathbf{\Gamma} - \mathbf{u}\mathbf{v}^\intercal$ (roughly the off-diagonal terms) be small enough compared to own-concavity (the diagonal terms), as reflected in Lemma 1. This, instead, calls for as small off-diagonal terms as possible, and thus low coordinate values for the vectors $\mathbf{u}, \mathbf{v}$.[10]

The following example proposes one particular construction of a rank-one shift, a *homogeneous* shift which appears in Ballester *et al.* (2006).

EXAMPLE 8 *Let* $\mathbf{\Gamma}$ *such that* $\gamma_{ii} > \gamma = \max\{\overline{\gamma}, 0\}$*, for all* $i = 1, ..., n$*, where* $\overline{\gamma} = \max\{\gamma_{ij} \mid i \neq j\}$*. Then, we can write the new signed interaction matrix* $\mathbf{\Gamma}' = \mathbf{\Gamma} - \gamma \mathbf{1}\mathbf{1}^{\intercal}$*, that is,* $\gamma'_{ij} = \gamma_{ij} - \gamma$*. For instance, in Example 6, the rank one shift corresponds to* $\gamma = 1$*.*

6 Discussion

In this paper we have explored a particular class of games where complementarities, although not present, may be induced by a linear signed transformation $\mathbf{T}$ applied to the original interaction matrix $\mathbf{\Gamma}$.

Both our emphasis on equilibrium existence, and the comparative statics for games with complementarities, are reminiscent of the literature on supermodular games (see Topkis (1979) and Vives (2005) for up-to-date results). The main differences between our model and this literature are the following.

First, the strategy space of a supermodular game is a bounded lattice. Here, instead, we deal with an un-bounded strategy space. Unlike with supermodular games, equilibrium existence is then an open question that calls for specific conditions on the moderation of the interaction pattern −the balance between own and cross effects. We also show that equilibrium existence and equilibrium uniqueness are two sides of same token. Beyond gaining insights into the exact working of positive feed-back loops in a population with general interaction modes, we believe that our results call for a word of caution. Namely, imposing an arbitrary bound on a strategy space need not be an innocuous modelling choice. Indeed, while this arbitrary bound solves equilibrium existence concerns, when the resulting lattice of equilibria does not boil down to a single outcome, the structure of this equilibrium lattice turns out to depend critically on the arbitrary choice of this upper bound.

Second, while we are able to characterize fully the (Katz) unique Nash equilibrium in games where complementarities need not be apparent, but are instead induced through an adequate transformation, our analysis is restricted to games with piece-wise linear best-responses and an unidimensional strategy space. The literature on supermodular games,

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[10] Given a vector $\mathbf{u} >> \mathbf{0}$, the best candidate vector $\mathbf{v}$ is such that the complementarity condition $\gamma_{ij} \leq u_i v_j$ binds.

instead, has a much wider scope. Future research should relax the linearity and unidimensionality assumptions and explore the connections between general (non-linear) games with hidden complementarities and the non-linear complementarity problem.

Our paper also belongs to the nascent literature on games on networks. We have already discussed some connections with the network public good game in Bramoullé and Kranton (2007), and with games with homogeneous shifted complementarities in Ballester *et al.* (2006). Broadly stated, these papers explore the role of network substitutabilities and complementarities in a complete information set up. Here, we establish connections between the two approaches that provide a more general perspective over this literature.

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Proof of Theorem 1. The result is a direct consequence of Theorem I in Pang (1979a). The requirement that the new interaction matrix $\mathbf{\Gamma T}$ is signed and moderate implies that it belongs to the class of *hidden Z-matrices*.

Proof of Lemma 1. This is a standard result in algebra. See Theorem 6.2.3 in Berman and Plemmons (1994).

Proof of Theorem 2. The conditions $\mathbf{u}_0 \geq \mathbf{0}$ and $s > \rho(\mathbf{G})$ allow first for a well-defined $\mathbf{u}$-Katz centrality $\mathbf{b}_{\mathbf{u}_0}(\mathbf{G}/s)$. Existence and uniqueness follows from combining Lemma 1 with Theorem 1.

We now show that this equilibrium results from applying the transformation $\mathbf{T}$ to the Katz-centrality vector. By taking $\mathbf{x}^*$, we verify that it satisfies the conditions of the linear complementarity problem (1). Note that in the linear complementarity problem, if $\mathbf{\Gamma}^{-1}\mathbf{u}_0$ is well-defined and non-negative then it must be a solution. The matrix $\mathbf{\Gamma}^{-1} = \mathbf{T}[s\mathbf{I} - \mathbf{G}]^{-1}$ is well-defined because $s > \rho(\mathbf{G})$. Moreover, the vector

$$\mathbf{\Gamma}^{-1}\mathbf{u}_0 = \mathbf{T}[s\mathbf{I} - \mathbf{G}]^{-1}\mathbf{u}_0 = (1/s)\mathbf{T}\mathbf{b}_{\mathbf{u}_0}(\mathbf{G}/s)$$

is non-negative by assumption. We conclude that it is the unique solution to the complementarity problem.

Proof of Corollary 1. By taking $\mathbf{T} = \mathbf{I}$, existence and uniqueness follow from Lemma 1 and Theorem 1.

Given the construction of the Nash equilibrium $\mathbf{x}^*$, we show that it satisfies the conditions of the linear complementarity problem. For every player i with $u_{0i} \leq 0$, we have that $x_i^* = 0$ and it is obvious that $u_{0i} - \sum_{j\in N}\gamma_{ij}x_j^* \leq 0$ and $x_i^*\left(u_{0i} - \sum_{j\in N}\gamma_{ij}x_j^*\right) = 0$. For every player i with $u_{0i} > 0$, we have that $x_i^* = (1/s)b_{i,\mathbf{u}_0^+}\left(G_{P(\mathbf{u}_0)}/s\right)$. It is easy to verify that the restricted vector $\mathbf{x}^*_{P(\mathbf{u}_0)} = (x_i^*)_{i\in P(\mathbf{u}_0)}$ solves the linear complementarity problem (1) restricted to the interaction matrix $\mathbf{\Gamma}_{P(\mathbf{u}_0)}$ and the vector $\mathbf{u}_0$. Thus, for all $i \in P(\mathbf{u}_0)$,

$$\begin{aligned} x_i^* &\geq 0, \\ u_{0i} - \sum_{j\in N}\gamma_{ij}x_j^* &= u_{0i} - \sum_{j\in P(\mathbf{u}_0)}\gamma_{ij}x_j^* \leq 0, \text{ and} \\ x_i^*\left(u_{0i} - \sum_{j\in N}\gamma_{ij}x_j^*\right) &= x_i^*\left(u_{0i} - \sum_{j\in P(\mathbf{u}_0)}\gamma_{ij}x_j^*\right) = 0. \end{aligned}$$

This means that the requirements for a solution to the unrestricted complementarity problem are met. We conclude that the vector $\mathbf{x}^*$ is the unique Nash equilbrium of the game $(\mathbf{u}_0, \mathbf{\Gamma})$.

Proof of Proposition 1. Existence and uniqueness follow from Lemma 1 and Theorem 1. To see this consider the signed transformation $\mathbf{T} = s\mathbf{I} - \mathbf{G}$. Then, $\mathbf{\Gamma T} = s^2\mathbf{I} - \mathbf{G}^2$ yields a unique equilibrium if and only if $s^2 > \rho\left(\mathbf{G}^2\right) = \rho\left(\mathbf{G}\right)^2$, where the last equality follows because $\mathbf{G} \geq \mathbf{0}$. Then, the condition $s > \rho\left(\mathbf{G}\right)$ is equivalent to $\mathbf{\Gamma T}$ being signed and moderate, and we may apply Theorem 1 to ensure existence and uniqueness of an equilibrium.

Again, the spectral condition $s > \rho\left(\mathbf{G}\right)$ and $\left(s\mathbf{I} - \mathbf{G}\right)\mathbf{b}_{\mathbf{u}_0}\left(\mathbf{G}/s\right) \geq 0$ guarantee that

$$\mathbf{\Gamma}^{-1}\mathbf{u}_0 = \left(s\mathbf{I} - \mathbf{G}\right)\left[s^2\mathbf{I} - \mathbf{G}^2\right]^{-1}\mathbf{u}_0 = \frac{1}{s^2}\left(s\mathbf{I} - \mathbf{G}\right)\mathbf{T}\mathbf{b}_{\mathbf{u}_0}\left(\mathbf{G}^2/s^2\right)$$

is well-defined and non-negative, and it is thus a solution to the associated linear complementarity problem.

Proof of Proposition 2. Let $\mathbf{\Psi} = \mathbf{\Gamma} - \mathbf{u}\mathbf{v}^\top$ be signed and moderate. Then, $\mathbf{\Psi}^{-1}$ is well-defined and non-negative. It is easy to prove that we can get the induced interaction matrix $\mathbf{\Psi} = \mathbf{T\Gamma}$, where

$$\mathbf{T} = \left(\mathbf{I} + \mathbf{\Psi}^{-1}\mathbf{u}\mathbf{v}^\intercal\right)^{-1} = \mathbf{I} - \frac{1}{1 + \mathbf{v}^\intercal\mathbf{\Psi}^{-1}\mathbf{u}}\mathbf{\Psi}^{-1}\mathbf{u}\mathbf{v}^\intercal.$$

The fact that $\mathbf{\Psi}$ is signed and moderate implies that $\mathbf{\Psi}^{-1} \geq \mathbf{0}$ and the second term is clearly nonnegative.

Thus, the transformation $\mathbf{T}$ is signed and Theorem 1 applies in order to get existence and uniqueness of an equilibrium: it is easy to see that $\mathbf{\Gamma}$ is moderate. The fact that $\mathbf{\Psi a} >> \mathbf{0}$ for some $\mathbf{a} \geq \mathbf{0}$, implies that, $\mathbf{\Gamma a} = \mathbf{\Psi a} + \mathbf{u}\mathbf{v}^\top\mathbf{a} >> \mathbf{0}$, i.e., $\mathbf{\Gamma}$ is a moderate interaction matrix.

By Corollary 1, we know that the game $\left(\mathbf{u}, \mathbf{\Psi} = \mathbf{\Gamma} - \mathbf{u}\mathbf{v}^\top\right)$ has a unique Nash equilibrium $\mathbf{y}^* = \mathbf{\Psi}^{-1}\mathbf{u}$. Consider the vector $\mathbf{x}^* = \mathbf{\Gamma}^{-1}\mathbf{u}$.[11] We prove that $\mathbf{x}^* \geq \mathbf{0}$ in order to qualify it as an equilibrium of the game $(\mathbf{u}, \mathbf{\Gamma})$. We can write

$$\mathbf{x}^*{=}\mathbf{T}\mathbf{\Psi}^{-1}\mathbf{u} = \frac{1}{1 + \mathbf{v}^\intercal\mathbf{\Psi}^{-1}\mathbf{u}}\mathbf{\Psi}^{-1}\mathbf{u},$$

which is non-negative given that $\mathbf{\Psi}^{-1} \geq \mathbf{0}$.

To prove the reciprocal result, note first that $\mathbf{u} - \mathbf{\Gamma x}^* \leq \mathbf{0}$, so that $\mathbf{\Psi x}^* \geq \left(1 - \mathbf{v}^\top\mathbf{x}^*\right)\mathbf{u} >> \mathbf{0}$. Thus, $\mathbf{\Psi}$ is moderate and yields a unique equilibrium by Lemma 1 and Corollary 1. Take $\mathbf{y}^*$ satisfying (6), that is, $\mathbf{y}^* = \left(\mathbf{I} - \mathbf{x}^*\mathbf{v}^\top\right)^{-1}\mathbf{x}^*$. This inverse is well-defined and non-negative because the spectral condition $1 > \rho\left(\mathbf{x}^*\mathbf{v}^\top\right) = \mathbf{v}^\top\mathbf{x}^*$ holds. Then, $\mathbf{y}^* \geq \mathbf{0}$. In order to show

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[11] The inverse of the matrix $\mathbf{\Gamma}$ exists because $\mathbf{\Gamma} = \mathbf{T}^{-1}\mathbf{\Psi}$ and both factor matrices $\mathbf{T}^{-1}$ and $\mathbf{\Psi}$ admit an inverse.

that $\mathbf{y}^*$ is a Nash equilibrium of $(\mathbf{u}, \mathbf{\Psi})$, we show that the linear complementarity conditions (1) hold. This immediately folows from the fact that $\mathbf{x}^*$ and $\mathbf{y}^*$ are proportional and that:

$$\begin{aligned}\mathbf{u} - \mathbf{\Psi}\mathbf{y}^* &= \left(1 + \mathbf{v}^\top \mathbf{y}^*\right)\mathbf{u} - \mathbf{\Gamma}\mathbf{y}^* \\ &= \left(1 + \mathbf{v}^\top \mathbf{y}^*\right)\left[\mathbf{u} - \mathbf{\Gamma}\mathbf{x}^*\right].\end{aligned}$$