Equality Matching: An Optimal Risk Sharing Institution

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Abstract

We consider a two-player game in which one player can take a costly action (i.e., to provide a favor) that is beneficial to the other. The game is infinitely repeated and each player is equally likely to be the one who can provide the favor in each period. In this context, equality matching is defined as a strategy in which each player counts the number of times she has given in excess of received and she gives if and only if this number has not reached an upper bound.

We show that the equality matching strategy is a symmetric subgame perfect equilibrium. Furthermore, we show that it remains an equilibrium when strategies are also required to be minimal complex. The two-state equality matching is the unique such strategy with two states, and so optimal within the class of the symmetric, subgame perfect, minimal complex strategies with two states. Furthermore, for any number of states n, the n-state equality matching maximizes the sum of long-run payoffs in the class of the strongly symmetric subgame perfect, minimally complex strategies with $m \leq n$ states. Thus, we rationalize equality matching as being an efficient way to achieve those properties.

This result is applied to risk sharing in village economies and used to rationalize the observed correlations between individual consumption and individual income and between present and past transfers across individuals.

1 Introduction

People that live in the villages of developing countries typically have a low and highly volatile income. In the absence of insurance and credit markets, informal institutions have developed there in order to allow for some risk sharing across individuals. In fact, people in village economies transfer a significant part of their income in order to assist those who have received a low income (see, for example, Fafchamps and Lund (2003)).

This practice of transferring part of one's income to assist others is an example of the equality matching form of sociality defined in Fiske (1992). In this form of behavior, each person maintains a balance, which increases one unit when she takes a costly action and decreases one unit when she benefits from a costly action taken by another person. This balance is then used to decide whether or not she should take a costly action again: she will take it if and only if the balance has not reached an upper bound. In the case of the village economies, people not only transfer part of their income to those in need (typically, referred to as a form of positive reciprocity), they also stop giving if the other never reciprocates, or does not reciprocate enough (a form of negative reciprocity). Indeed, as Fafchamps and Lund (2001, p. 28) have shown, there is a significant negative correlation between current and past transfers received by individuals in their sample.

Why do we observe equality matching? Is there a sense in which this form of behavior is optimal? While one can easily explain the positive reciprocity aspect of equality matching through repeated interaction, is it the case that we can understand both its positive and negative aspect as being simultaneously part of an optimal equilibrium behavior?

In this paper, we provide an answer to these questions. We consider an infinitely repeated two-player game in which one player can take a costly action (i.e., provide a favor) that is beneficial to the other, and in which each player is equally likely to be the one who can provide the favor in every period.[1] Several authors have pointed out that many real life institutions are self-enforcing, treat individuals symmetrically, cannot be simplified, and their rules are simple to understand. Following their work, we define a social institution as a repeated game strategy with those properties. Then, we show that the equality matching strategy satisfies those properties in an optimal way: the welfare of each player is at least as high under the equality matching strategy as under any other social institution of the same complexity. Hence, in this sense, equality matching is an optimal social institution.

In the particular case in which the costly action consists of transferring part of an individual's endowment, equality matching implies a particular pattern of individual consumption and transfers that is consistent with observed correlations in village economies. In fact, it implies some risk sharing, which is not complete due to its negative reciprocity aspect. Moreover, it implies a positive correlation between individual consumption and current and lagged individual income (documented in Townsend (1994), among others) and a negative correlation between current and past transfers among individuals (reported, for instance, in Fafchamps and Lund (2003) and La Ferrara (2003)). The advantage of our theory compared to that of Kocherlakota (1996) is that it has the potential to generate stronger correlations between individual consumption and individual income, current and lagged, and between current and past transfers. In particular, it generates non-zero correlations even if consumers are extremely patient. This is in contrast to the main results of Kocherlakota (1996) since: first, if players are sufficiently patient, they predict that those correlations equal zero; second, as Koeppl (2006) and Rincón-Zapatero and Santos (2006) have shown, this can still be the case even if players are sufficiently impatient.

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[1] Thus, this game is a symmetric repeated dictator game, where by "symmetric" we mean that in every period each player is equally likely to be the dictator. Naturally, we assume that the benefit is higher than the cost of the action.

2 The Model

There are two players that interact in every period $t \in \mathbb{N}$. In every period, one of them can provide a favor to the other; we assume that this is decided by nature, in a way that each player has in every period a 1/2 probability of being the one who can provide the favor.

When a player provides a favor, he suffers a utility cost $d > 0$, and the player receiving it obtains a positive utility $u > 0$. If the favor is not provided, then both players receive zero utility. We assume that favors are efficient in the sense that their benefit exceeds their cost. That is, we assume that $u > d$.

Let $N = \{1, 2\}$ stand for the set of players, $\Omega = \{1, 2\}$ for the set of states of nature, and $A = \{P, NP\}$ for the set of possible actions. We make the convention that when the state of nature equals 1, only player 1 can provide a favor, and so he chooses an action from the set A; similarly, when the state of nature equals 2, player 2 is the one who can provide the favor. The payoffs, which players receive period-wise, and which depend on the state of nature and on the choice made by the player who can provide the favor, are summarized in the following table:

$\omega \backslash a$	P	NP
1	$-d, u$	0,0
2	$u, -d$	0,0

Table 1: Stage Game Payoffs

We denote the period-wise payoffs as $u_i(\omega, a)$.

We describe the behavior of each player in the repeated game by an automaton. An *automaton* for player i is a triple $I_i = ((S_i, \bar{s}_i), T_i, B_i)$ where: S_i is a set of *states*; $\bar{s}_i \in S_i$ is the *initial state*; $T_i : \Omega \times S_i \times A \to S_i$ is a *transition function*; and $B_i : S_i \to A$ is a *behavior function.*

A pair of individual automata $I = (I_1, I_2)$, or for short, an automaton, together with a sequence of states of nature $\boldsymbol{\omega} = \{\omega_k\}_{k=1}^{\infty} \subseteq \Omega$ induce a sequence of actions

$\mathbf{a}(I, \boldsymbol{\omega}) = \{a_k\}_{k=1}^{\infty} \subseteq A$ in the following way: $a_1 = B_{\omega_1}(\bar{s}_{\omega_1})$, and $a_k = B_{\omega_k}(s_{\omega_k}^k)$, where $s_i^k = T_i(s_i^{k-1}, a_{k-1})$, for both $i = 1, 2$.[2] Let $\mathbf{a}_k(I, \boldsymbol{\omega})$ denote the kth coordinate of $\mathbf{a}(I, \boldsymbol{\omega})$ for all $k \in \mathbb{N}$.

Each player's payoff in the repeated game depends on the payoff he receives in all periods, in the following way: first, for all $i = 1, 2$, define

$$U_i^{\delta}(I, \boldsymbol{\omega}) = (1 - \delta) \sum_{k=1}^{\infty} \delta^{k-1} u_i(\omega_k, \mathbf{a}_k(I, \boldsymbol{\omega})), \tag{1}$$

Second, let $\boldsymbol{\Omega} = \Omega \times \Omega \times \cdots$ and μ be product measure on $\boldsymbol{\Omega}$. Then, the discounted payoff of an automaton I for player i, $i = 1, 2$, is

$$U_i^{\delta}(I) = \int_{\boldsymbol{\Omega}} U_i^{\delta}(I, \boldsymbol{\omega}) \mathrm{d}\mu(\boldsymbol{\omega}). \tag{2}$$

We also define the long-run payoff of an automaton to be

$$U_i(I) = \lim_{\delta \to 1} U_i^{\delta}(I) \tag{3}$$

for all players $i = 1, 2$ and automaton I.

For all automata $I = ((S, \bar{s}), T, B)$ and $s \in S$, define $R(s) = \{s' \in S : s' = T(\omega, s, a)$ for some $(\omega, a)\}$ be the set of transitions emanating from s.

Our complexity ordering $\succeq^c$ is defined as follows. Let $I = ((S, \bar{s}), T, B)$ and $I' = ((S', \bar{s}'), T', B')$ be two automata. We say that I is at least as complex as I', $I \succeq^c I'$, if

1. $|S| \geq |S'|$ and
2. there is a subset $\hat{S}$ of S and a bijection $\sigma : S' \to \hat{S}$ such that $|R(s')| \leq |R(\sigma(s'))|$ for all $s' \in S'$,

Furthermore, if any of the above inequalities hold strictly, then I is more complex than I', denoted as $I \succ^c I'$.

Player i has preferences $\succeq_i$ over pairs of automata. Such preferences lexicographically compares the payoffs and the complexity that any such pair induces. Thus, we assume that $(I_1', I_2') \succ_i (I_1, I_2)$ if either $U_i(I_1, I_2) > U_i(I_1', I_2')$ or $U_i(I_1, I_2) = U_i(I_1', I_2')$ and $I_i \succ^c I_i'$.

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[2] Recall that player i is the producer in period k if $\omega_k = i$, for all $i = 1, 2$.

3 Equality Matching

Formally, an *equality matching automaton* $I_M = (I_1^M, I_2^M)$ *with a threshold* $M \in \mathbb{N}$ is defined as follows: the set of states is

$$S_1^M = S_2^M = S_M = \{0, \ldots, M\}, \tag{4}$$

and the initial state is $\bar{s}_M \in S_M$. The state space is interpreted as representing player 1's balance and $\bar{s}_M$ as his initial balance. Thus, $M - \bar{s}_M$ represents player 2's balance.

Player 1's behavior function is defined by:

$$B_1^M(m) = \begin{cases} P \text{ if } m < M, \\ NP \text{ otherwise.} \end{cases} \tag{5}$$

Similarly, Player 2's behavior function is defined as follows:

$$B_2^M(m) = \begin{cases} P \text{ if } m > 0, \\ NP \text{ otherwise.} \end{cases} \tag{6}$$

Intuitively, any player provides a favor if and only if the other player has a positive balance, which occurs if and only if his own balance has not reached the upper bound.

The transition function $T_1^M = T_2^M = T_M : \Omega \times S_M \times A \to S_M$ is defined by

$$T_M(1, m, a) = \begin{cases} m+1 & \text{if} \quad m < M \text{ and } a = P, \\ M & \text{if} \quad m = M \text{ and } a = NP, \\ 0 & \text{if} \quad m = 0 \text{ and } a = NP, \\ m-1 & \text{if} \quad 0 < m < M \text{ and } a = NP, \\ M-1 & \text{if} \quad m = M \text{ and } a = P, \end{cases} \tag{7}$$

when the state of nature is 1. Similarly, we have

$$T_M(2, m, a) = \begin{cases} m-1 & \text{if} \quad m > 0 \text{ and } a = P, \\ 0 & \text{if} \quad m = 0 \text{ and } a = NP, \\ M & \text{if} \quad m = M \text{ and } a = NP, \\ m+1 & \text{if} \quad 0 < m < M \text{ and } a = NP, \\ 1 & \text{if} \quad m = 0 \text{ and } a = P, \end{cases} \tag{8}$$

The interpretation is as follows. Whenever player 1 provides a favor, her balance increases by 1 unit, except when this balance has reached the upper bound M. In the latter case, the behavior function recommends that no favor should be provided and in this case player 1's balance remains at M. Similarly, whenever she receives a favor, her balance decreases by 1 unit, except when it has reached 0. Since player 2's balance is just $M - s$, the latter case occurs exactly when player 2's balance has reached the upper bound. There is punishment in the above automaton since if any player deviates (either from P or from NP), then his balance decreases one unit. The definition of I_M does describe equality matching in the sense that each player takes costly actions that benefit the other, but will stop doing so if this other player does not reciprocate enough.

4 Equilibria

A pair of automata $I = (I_1, I_2)$ is a *Nash equilibrium with complexity costs* (NEC) if for each player i and automaton I'_i, $I \succeq (I'_i, I_{-i})$. A pair of automata I is a *subgame perfect equilibrium with complexity* (PEC) if I is a subgame perfect equilibrium and a NEC.

4.1 Equality Matching is a perfect equilibrium with complexity

In this section, we first show that every EM automaton is a PEC for sufficiently large discount factor (Proposition 1). Then, we characterize those EM automata that are a PEC for a fixed discount factor in terms of two incentive equations (Proposition 2). Finally, we use this characterization to establish the following monotonicity property of equilibrium EM automata: there is a threshold level M^* such that all EM automata with a smaller or equal threshold level are PEC while those with a larger threshold level are not (Proposition 3).

Our first proposition is the following asymptotic result.

Proposition 1 *For all $M \in \mathbb{N}$, there is $\delta^* \in (0,1)$ such that every equality matching automaton I_M is a PEC.*

Now consider the case of a fixed discount factor. The following propositions characterize the set of equilibrium EM automata. Let V_m be the expected discounted payoff of player 1 when state $m \in S_M$ is the initial state.

Proposition 2 *The EM automaton I_M is a PEC if and only if $\delta(V_1 - V_0) > (1-\delta)d$ and $\min_{m\in\{2,\dots,M\}} \delta(V_m - V_{m-2}) > (1-\delta)d$.*

For the following result, it is convenient to make explicit the dependence of V on the threshold M. Thus, we write $V_M = (V_0^M, \dots, V_M^M)$ for player 1's expected continuation payoff under I_M. Define M^* to be the maximal M such that

$$\min\{\delta(V_1^M - V_0^M), \min_{m\in\{2,\dots,M\}} \delta(V_m^M - V_{m-2}^M)\} > (1-\delta)d.$$

Proposition 3 *The EM automaton I_M is a PEC if and only if $M \leq M^*$. Furthermore, M^* converges to infinity as δ converges to one.*

4.2 Proof of the equilibrium properties of equality matching

In order to analyze the equilibrium properties of the EM automata, it is convenient to consider the expected discounted payoff $V_m(\delta)$ of player 1 associated with each state $m \in S_M$. It is clear from the definition of the EM automata that $(V_0(\delta), \dots, V_M(\delta))$ satisfy:

$$\begin{aligned}
V_M(\delta) &= (1-\delta)\frac{u}{2} + \delta\frac{V_M(\delta) + V_{M-1}(\delta)}{2}, \\
V_m(\delta) &= (1-\delta)\frac{u-d}{2} + \delta\frac{V_{m+1}(\delta) + V_{m-1}(\delta)}{2} \quad \text{for all } 0 < m < M, \\
V_0(\delta) &= -(1-\delta)\frac{d}{2} + \delta\frac{V_1(\delta) + V_0(\delta)}{2}.
\end{aligned} \tag{9}$$

This system can be rearranged in two convenient ways. The first, used below in the proof of Lemma 5, is:

$$\begin{aligned}\frac{V_M(\delta) - V_{M-1}(\delta)}{1-\delta} &= \frac{u}{2-\delta} - \frac{2V_{M-1}(\delta)}{2-\delta}, \text{ and} \\ \frac{V_m(\delta) - V_{m-1}(\delta)}{1-\delta} &= \frac{u-d}{2-\delta} + \frac{\delta}{2-\delta}\frac{V_{m+1}(\delta) - V_m(\delta)}{1-\delta} - \frac{2V_{m-1}(\delta)}{2-\delta}\end{aligned} \tag{10}$$

for all $0 < m < M$. See Appendix A.1 for a proof.

The second way to rearrange 9 is as follows. Defining $\gamma_m = V_m - V_{m-1}$ for all $1 \leq m \leq M$, it follows that

$$2\gamma_1 - \delta\gamma_2 = (1-\delta)u, \tag{11}$$

$$-\delta\gamma_{m-1} + 2\gamma_m - \delta\gamma_{m+1} = 0 \text{ for all } 1 < m < M, \tag{12}$$

$$-\delta\gamma_{M-1} + 2\gamma_M = (1-\delta)d. \tag{13}$$

This system of equations can be written as follows,

$$\begin{pmatrix} 2 & -\delta & 0 & 0 & \cdots & 0 & 0 \\ -\delta & 2 & -\delta & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -\delta \\ 0 & 0 & 0 & 0 & \cdots & -\delta & 2 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{M-1} \\ \gamma_M \end{pmatrix} = \begin{pmatrix} (1-\delta)u \\ 0 \\ \vdots \\ 0 \\ (1-\delta)d \end{pmatrix} \tag{14}$$

or more compactly as $A\gamma = v$.

The matrix A is a tridiagonal matrix and its inverse can be obtained in the following way (see Dow (2003, Section 3.1)): Letting

$$r_1 = \frac{1 + \sqrt{1-\delta^2}}{\delta} \tag{15}$$

$$r_2 = \frac{1 - \sqrt{1-\delta^2}}{\delta} \tag{16}$$

$$P_{ij} = \frac{(r_1^i - r_2^i)(r_1^{M+1-j} - r_2^{M+1-j})}{\delta(r_1 - r_2)(r_1^{M+1} - r_2^{M+1})} \text{ and} \tag{17}$$

$$Q_{ij} = \frac{(r_1^j - r_2^j)(r_1^{M+1-i} - r_2^{M+1-i})}{\delta(r_1 - r_2)(r_1^{M+1} - r_2^{M+1})}, \tag{18}$$

then

$$A_{ij}^{-1} = \begin{cases} P_{ij} & \text{if} \quad i \leq j, \\ Q_{ij} & \text{if} \quad j \leq i. \end{cases} \tag{19}$$

Therefore,

$$\frac{\delta\gamma_m}{1-\delta} = \frac{r_1^{M+1-m} - r_2^{M+1-m}}{r_1^{M+1} - r_2^{M+1}} u + \frac{r_1^m - r_2^m}{r_1^{M+1} - r_2^{M+1}} d \tag{20}$$

for all $1 \le m \le M$.

Lemma 1 *The number r_1 and r_2 satisfy the following properties: $r_1 > 1$, $0 < r_2 < 1$ and*

$$\frac{1-r_2}{r_1-r_2} r_1^2 + \frac{r_1-1}{r_1-r_2} r_2^2 > 1.$$

Proof. It is clear that $r_1 > 1$ and $r_2 > 0$. Furthermore, we have that $r_2 < 1$. In fact, $r_2 < 1 \Leftrightarrow 1-\delta < \sqrt{1-\delta^2} \Leftrightarrow (1-\delta)^2 < 1-\delta^2 \Leftrightarrow -2\delta(1-\delta) < 0$, which holds since $\delta \in (0,1)$.

Regarding the remaining inequality, we have that:

$$r_1 - r_2 = \frac{2\sqrt{1-\delta^2}}{\delta}, \tag{21}$$

$$1 - r_2 = \frac{-(1-\delta) + \sqrt{1-\delta^2}}{\delta}, \tag{22}$$

$$r_1 - 1 = \frac{1-\delta+\sqrt{1-\delta^2}}{\delta}, \tag{23}$$

$$\frac{1-r_2}{r_1-r_2} = \frac{1}{2} - \frac{1-\delta}{2\sqrt{1-\delta^2}}, \tag{24}$$

$$\frac{r_1-1}{r_1-r_2} = \frac{1}{2} + \frac{1-\delta}{2\sqrt{1-\delta^2}}. \tag{25}$$

$$\tag{26}$$

Thus,

$$\frac{1-r_2}{r_1-r_2} r_1^2 + \frac{r_1-1}{r_1-r_2} r_2^2 = \frac{r_1^2+r_2^2}{2} - \frac{1-\delta}{2\sqrt{1-\delta^2}}(r_1^2 - r_2^2).$$

We have that

$$r_1^2 = \frac{1+2\sqrt{1-\delta^2}+1-\delta^2}{\delta^2}, \tag{27}$$

$$r_2^2 = \frac{1-2\sqrt{1-\delta^2}+1-\delta^2}{\delta^2}, \tag{28}$$

$$\frac{r_1^2+r_2^2}{2} = \frac{2-\delta^2}{\delta^2}, \tag{29}$$

$$r_1^2 - r_2^2 = \frac{4\sqrt{1-\delta^2}}{\delta^2}, \tag{30}$$

$$\frac{1-\delta}{2\sqrt{1-\delta^2}}(r_1^2 - r_2^2) = \frac{2(1-\delta)}{\delta^2}. \tag{31}$$

Hence,

$$\frac{1-r_2}{r_1-r_2}r_1^2+\frac{r_1-1}{r_1-r_2}r_2^2=\frac{2-\delta^2-2+2\delta}{\delta^2}=\frac{2}{\delta}-1>1$$

since $2/\delta>2$. ■

The above formula for $\delta\gamma_m/(1-\delta)$, $m\in S_M$, and the fact that $r_1>1>r_2>0$ immediately imply that player 1 (and symmetrically, player 2) gains by moving to an higher state. In other words, the marginal utility of accumulating a balance is positive.

Lemma 2 *For all* $1\leq m\leq M$, $V_m>V_{m-1}$.

The following lemma consider limit case of $\delta=1$.

Lemma 3 *For all* $m\in\{1,\ldots,M\}$,

$$\lim_{\delta\to 1}\frac{\delta(V_m(\delta)-V_{m-1}(\delta))}{1-\delta}=\frac{M+1-m}{M+1}u+\frac{m}{M+1}d.$$

Proof. Let $\{\delta_k\}_{k=1}^{\infty}$ be such that $\delta_k\to 1$ and let $\{r_{1,k}\}_{k=1}^{\infty}$ and $\{r_{2,k}\}_{k=1}^{\infty}$ be the corresponding sequences for r_1 and r_2. For convenience, let $f(r)=r^{M+1-m}$ and $g(r)=r^{M+1}$. Then, by the generalized mean value theorem (see Rudin (1976, Theorem 5.9, p. 107)) for all k there exists $r_k\in(r_{2,k},r_{1,k})$ such that

$$\frac{r_{1,k}^{M+1-m}-r_{2,k}^{M+1-m}}{r_{1,k}^{M+1}-r_{2,k}^{M+1}}=\frac{f(r_{1,k})-f(r_{2,k})}{g(r_{1,k})-g(r_{2,k})}=\frac{f'(r_k)}{g'(r_k)}=\frac{(M+1-m)r_k^{M-m}}{(M+1)r_k^M}.$$

Since $r_{2,k}<r_k<r_{1,k}$ for all k and $r_{1,k},r_{2,k}\to 1$, it follows that $r_k\to 1$ and so $(r_{1,k}^{M+1-m}-r_{2,k}^{M+1-m})/(r_{1,k}^{M+1}-r_{2,k}^{M+1})$ converges to $(M+1-m)/(M+1)$.

Similarly, for all k there exists $\hat{r}_k\in(r_{2,k},r_{1,k})$ such that

$$\frac{r_{1,k}^m-r_{2,k}^m}{r_{1,k}^{M+1}-r_{2,k}^{M+1}}=\frac{m\hat{r}_k^{M-m}}{(M+1)\hat{r}_k^M}$$

and so $(r_{1,k}^m-r_{2,k}^m)/(r_{1,k}^{M+1}-r_{2,k}^{M+1})$ converges to $m/(M+1)$, again since $\hat{r}_k\to 1$. Thus,

$$\frac{\delta_k\gamma_{m,k}}{1-\delta_k}=\frac{r_{1,k}^{M+1-m}-r_{2,k}^{M+1-m}}{r_{1,k}^{M+1}-r_{2,k}^{M+1}}u+\frac{r_{1,k}^m-r_{2,k}^m}{r_{1,k}^{M+1}-r_{2,k}^{M+1}}d\to\frac{M+1-m}{M+1}u+\frac{m}{M+1}d.$$

■

Lemma 3 implies that the incentive inequalities will hold strictly for sufficiently large δ. This fact is useful to establish Proposition 1.

Lemma 4 *Suppose that I satisfies the following property: for all $i \in \{1, 2\}$, $s_i \in S_i$ and $a \neq B_i(s_i)$,*

$$(1-\delta)u_i(i, B_i(s_i)) + \delta V^i_{\tilde{s}_i} > (1-\delta)u_i(i, a) + \delta V^i_{s'_i} \tag{32}$$

where $\tilde{s}_i = T_i(i, s_i, B_i(s_i))$ and $s'_i = T_i(i, s_i, a)$.

Let I'_i be such that there exists $k \in \mathbb{N}$ and $\omega^k \in \Omega^k$ such that $\mathbf{a}((I'_i, I_{-i}), \omega^k) \neq \mathbf{a}(I, \omega^k)$. Then, $U^\delta_i(I) > U^\delta_i(I'_i, I_{-i})$.

Proof. Let n be the minimal k such that there exists $\omega^k \in \Omega^k$ satisfying $\mathbf{a}_k((I'_i, I_{-i}), \omega^k) \neq \mathbf{a}_k(I, \omega^k)$. Then, the path of actions coincide in the two automata until period n, and so

$$U^\delta_i(I) - U^\delta_i(I'_i, I_{-i}) = \sum_{\tilde{\omega}^n \in \Omega^n} \frac{1}{2^n} \left(U^\delta_i(I|\tilde{\omega}^n) - U^\delta_i(I'_i, I_{-i}|\tilde{\omega}^n) \right).$$

By condition (32), it follows that $U^\delta_i(I|\omega^n) - U^\delta_i(I'_i, I_{-i}|\omega^n) > 0$. Furthermore, condition (32) also implies that I is a subgame perfect equilibrium and so $U^\delta_i(I|\tilde{\omega}^n) - U^\delta_i(I'_i, I_{-i}|\tilde{\omega}^n) \geq 0$ for all $\tilde{\omega}^n \in \Omega^n$. Thus, $U^\delta_i(I) - U^\delta_i(I'_i, I_{-i}) > 0$. ■

Finally, we turn to the proof of Proposition 1, which states that for all $M \in \mathbb{N}$, there is $\delta^* \in (0, 1)$ such that every EM automaton I_M is a PEC.

Proof of Proposition 1. It follows from Lemma 3 that condition (32) of Lemma 4 is satisfied.

We next show that if I_1 is such that $|S_1| < |S_M|$, then $U_1(I_1, I^M_2) < U_1(I_M)$. By Lemma 4, it suffices to find $k \in \mathbb{N}$ and $\omega^k \in \Omega^k$ such that $\mathbf{a}(I_M, \omega^k) \neq \mathbf{a}((I_1, I^M_2), \omega^k)$. Suppose in order to reach a contradiction that $\mathbf{a}(I_M, \omega^k) = \mathbf{a}((I_1, I^M_2), \omega^k)$ for all $k \in \mathbb{N}$ and $\omega^k \in \Omega^k$. Let $m = \bar{s}_2 \in \{0, \ldots, M\}$, $l = m + 1 + M$ and ω^l defined by

$$\omega^l_t = \begin{cases} 2 & \text{if} \quad t \leq m, \\ 1 & \text{if} \quad t > m. \end{cases}$$

Since $\mathbf{a}(I_M, \omega^l) = \mathbf{a}((I_1, I^M_2), \omega^l)$, the sequence $\{s^2_t\}^l_{t=1}$ is the same under I_M and under (I_1, I^M_2). Applying the definition of I_M, we obtain the following sequence of actions and player 2's states.

t	1	$\cdots$	$m+1$	$m+2$	$\cdots$	$l-1$	l
s_t^2	m	$\cdots$	0	1	$\cdots$	$M-1$	M
ω_t^l	2	$\cdots$	1	1	$\cdots$	1	1
$\mathbf{a}_t$	P	$\cdots$	P	P	$\cdots$	P	NP

Since $|S_1| < M+1$ and there are $M+1$ elements in $\{s_t^1\}_{t=m+1}^l$ (recall that $l = m+1+M$), it must be that $s_j^1 = s_{j+r}^1$ for some $j \in \{m+1,\ldots,m+M\}$ and $r > 0$ such that $j+r \in \{m+2,\ldots,m+1+M\}$. Since $\omega_j^l = \omega_{j+r}^l = 1$ and $B_1(s_j^1) = B_1(s_{j+r}^1)$, then $s_{j+1}^1 = s_{j+r+1}^1$. By induction, we obtain $s_{l-r}^1 = s_l^1$ and so $P = \mathbf{a}_{l-r}(I_M,\omega^l) = \mathbf{a}_{l-r}((I_1,I_2^M),\omega^l) = B_1(s_{l-r}^1) = B_1(s_l^1) = \mathbf{a}_l((I_1,I_2^M),\omega^l) = \mathbf{a}_l(I_M,\omega^l) = NP$, a contraction. This contradiction establishes that player 1 cannot save on states.

Thus, it remains to show that player 1 cannot save on the transitions. Note that $|R_1(s)| = 2$ for all $s \in S^M$. In order to reach a contradiction suppose that there exists I_1 such that $U_1(I_1,I_2^M) = U_1(I_M)$, $|S_1| = M+1$ and $|R_1(s)| = 1$ for some $s \in S_1$. Thus, there exists $s' \in S_1$ such that $T_1(\omega,s,a) = s'$ for all $(\omega,a) \in \Omega \times A$.

Let $k \in \mathbb{N}$ be such that player 1's state in period k is s. Since $U_1(I_1,I_2^M) = U_1(I_M)$, Lemma 4 implies that $\mathbf{a}(I_M,\omega^l) = \mathbf{a}((I_1,I_2^M),\omega^l)$ for all $l \in \mathbb{N}$ and $\omega^l \in \Omega^l$. Note that if state s could not be reached, then player 1 could drop it and obtain the same payoff with M states, which, as we have seem, it is not possible. Hence, such k exists. Let $\tilde{\omega}^{k-1}$ be such that the state of player 1 in period k is s. Let $m \in \{0,\ldots,M\}$ be such that $s_k^2 = M-m$. Let $r = m$ if $m > 0$ and $r = 1$ if $m = 0$ and define ω^{k+r} and $\hat{\omega}^{k+r}$ so that $\omega_k^{k+r} = 1$, $\hat{\omega}_k^{k+r} = 2$ and

$$\omega_t^{k+r} = \hat{\omega}_t^{k+r} = \begin{cases} \tilde{\omega}_t^{k-1} & \text{if} \quad t < k, \\ 1 & \text{if} \quad t > k. \end{cases}$$

For a fixed $\breve{\omega}^{k+r} \in \{\omega^{k+r},\hat{\omega}^{k+r}\}$, the sequence $\{\breve{s}_t^2\}_{t=k+1}^{k+r}$ is the same under I_M and under (I_1,I_2^M) since $\mathbf{a}(I_M,\breve{\omega}^{k+r}) = \mathbf{a}((I_1,I_2^M),\breve{\omega}^{k+r})$. Applying the definition of I_2^M and that of ω^{k+r}, and using the fact that $s' = T_1(\omega,s,a)$ for all $(\omega,a) \in \Omega \times A$, we obtain the following sequence $\{(s_t^1,s_t^2,\mathbf{a}_t(I_M,\omega^{k+r}))\}_{t=k}^{k+r}$ of states and actions.

t	k	$k+1$	$\cdots$	$k+r$
s_t^1	s	s'	$\cdots$	s_{k+r}^1
s_t^2	$M-m$	$M-m+1$	$\cdots$	M
ω_t^{k+r}	1	1	$\cdots$	1
$\mathbf{a}_t(I_M, \omega^{k+r})$	P	P	$\cdots$	NP

Note also that under $\hat{\omega}^{k+r}$, $\hat{s}_{k+1}^2 = M - m + 1 - \theta$ for some $\theta \in \{1, 2\}$. In fact, if $m = M$, then $B_2(0) = NP$ (recall that $\hat{\omega}_k^{k+r} = 2$) and so $\hat{s}_{k+1}^2 = 0 = 1 - \theta$ with $\theta = 1$; and if $m < M$, then $B_2(M-m) = P$ and so $\hat{s}_{k+1}^2 = M - m - 1 = M - m + 1 - \theta$ with $\theta = 2$. Hence, applying the definition of I_2^M and that of $\hat{\omega}^{k+r}$, and using the fact that $s' = T_1(\omega, s, a)$ for all $(\omega, a) \in \Omega \times A$, we obtain the following sequence $\{(\hat{s}_t^1, \hat{s}_t^2, \mathbf{a}_t(I_M, \omega^{k+r}))\}_{t=k}^{k+r}$ of states and actions.

t	k	$k+1$	$\cdots$	$k+r$
$\hat{s}_t^1$	s	s'	$\cdots$	s_{k+r}^1
$\hat{s}_t^2$	$M-m$	$M-m+1-\theta$	$\cdots$	$M-\theta$
$\hat{\omega}_t^{k+r}$	2	1	$\cdots$	1
$\mathbf{a}_t(I_M, \hat{\omega}^{k+r})$	P or NP	P	$\cdots$	P

Note that $s_{k+r}^1 = \hat{s}_{k+r}^1$ since $s_{k+1}^1 = \hat{s}_{k+1}^1 = s'$, $\omega_t^{k+r} = \hat{\omega}_t^{k+r}$ and $\mathbf{a}_t(I_M, \omega^{k+r}) = \mathbf{a}_t(I_M, \hat{\omega}^{k+r})$ for all $k+1 \leq t \leq k+r-1$. It then follows that $NP = \mathbf{a}_{k+r}(I_M, \omega^{k+r}) = \mathbf{a}_{k+r}((I_1, I_2^M), \omega^{k+r}) = B_1(s_{k+r}^1) = \mathbf{a}_{k+r}((I_1, I_2^M), \tilde{\omega}^{k+r}) = \mathbf{a}_{k+r}(I_M, \tilde{\omega}^{k+r}) = P$, a contraction. This contradiction establishes that player 1 cannot save on transitions.

Since a similar property holds symmetrically for player 2, the proof is complete.

■

Proposition 2 states that the EM automaton I_M is a PEC if and only if $\delta(V_1 - V_0) > (1-\delta)d$ and $\min_{m \in \{2, \ldots, M\}} \delta(V_m - V_{m-2}) > (1-\delta)d$.

Proof of Proposition 2. Sufficiency follows from the proof of Proposition 1.

Regarding the necessity part, note that if $\min\{\delta(V_1 - V_0), \min_{m \in \{2, \ldots, M\}} \delta(V_m - V_{m-2})\} < (1-\delta)d$, then I_M is not subgame perfect. Thus, we may assume that $\min\{\delta(V_1 - V_0), \min_{m \in \{2, \ldots, M\}} \delta(V_m - V_{m-2})\} = (1-\delta)d$.

If $\delta(V_1 - V_0) = (1-\delta)d$, define I_1 by $T_1 = T_1^M$ and $B_1 = B_1^M$ except that $T_1(\omega, 0, a) = 0$ for all (ω, a) and $B_1(0) = NP$. Similarly, if $\delta(V_m - V_{m-2}) = (1-\delta)d$, for some $m \in \{0, \ldots, M\}$, define I_1 by $T_1 = T_1^M$ and $B_1 = B_1^M$ except that $T_1(\omega, m-1, a) = m-2$ for all (ω, a) and $B_1(m-1) = NP$. Then, the number of transitions is smaller in I_1 and (I_1, I_2^M) yield the same payoff as I_M to player 1. Indeed, if player 1 were to deviate just once, he would get the same payoff by assumption, and this implies, by induction, that the above deviation yields the same payoff. Hence, I_M is not a PEC. ■

Lemma 5 provides a sufficient condition for the marginal gain of moving to an higher state to be decreasing. The sufficient condition say that it is worthwhile for player 1 to give in state $M-1$. Hence, it follows from the lemma that if player 1 is willing to give at state $M-1$, then he is willing to give at all states $m \geq 2$ (state 1 is different since the incentive equation is $\delta(V_1 - V_0) \geq (1-\delta)d$, while for state $2 \leq m \leq M-1$ is $\delta(V_m - V_{m-2}) \geq (1-\delta)d$).

Lemma 5 *If $\delta(V_M - V_{M-1}) \geq (1-\delta)d$, then $\gamma_m < \gamma_{m-1}$ for all $1 \leq m \leq M$.*

Proof. For all $1 \leq m \leq M$, let $D_m = (V_m - V_{m-1})/(1-\delta) = \gamma_m/(1-\delta)$. Clearly, it suffices to show that $D_m - D_{m-1} < 0$.

We establish the above claim by induction. From (10), it follows that

$$D_M - D_{M-1} = \frac{d}{2-\delta} - \frac{\delta}{2-\delta} D_M - \frac{2(1-\delta)}{2-\delta} D_{M-1}.$$

Since $D_{M-1} > 0$ by Lemma 2 and, by assumption, $\delta D_M > d$, it follows that

$$(2-\delta)(D_{M-1} - D_M) \;= -d + \delta D_M + 2(1-\delta)D_{M-1} > 2(1-\delta)D_{M-1} > 0.$$

Hence, $(2-\delta)(D_{M-1} - D_M) > 0$ and so $D_M - D_{M-1} < 0$.

Suppose that $D_{m+1} - D_m < 0$. From (10), it follows that

$$D_m - D_{m-1} = \frac{\delta}{2-\delta}(D_{m+1} - D_m) - \frac{2\gamma_{m-1}}{2-\delta} < 0$$

since $\gamma_{m-1} > 0$ by Lemma 2. Thus, the lemma follows by induction. ■

It follows from Lemma 5 that when $\delta(V_M - V_{M-1}) \geq (1-\delta)d$, then $\delta(V_1 - V_0) > (1-\delta)d$ and $\delta(V_m - V_{m-2}) = \delta(\gamma_m + \gamma_{m-1}) > (1-\delta)d$. Hence, we obtain the following sufficient condition for I_M to be a PEC.

Corollary 1 *If* $\delta(V_M - V_{M-1}) \geq (1-\delta)d$*, then* I_M *is a PEC.*

Define M^* to be the maximal M such that

$$\min\{\delta(V_1^M - V_0^M), \delta(\min_{m\in\{2,\dots,M\}} V_M^m - V_{m-2}^M)\} > (1-\delta)d.$$

The following lemma shows that M^* is well defined.

Lemma 6 *For all* $\delta \in (0,1)$*, there is* $\overline{M} \in \mathbb{N}$ *such that* $\delta(\min_{m\in\{2,\dots,M\}} V_m^M - V_{m-2}^M) < (1-\delta)d$ *for all* $M \geq \overline{M}$*.*

Proof. Consider first the case where M is even. From (20) it follows that

$$\frac{\delta\gamma_{M/2}}{1-\delta} = \frac{r_1^{-M/2} - r_2^{M/2+1}/r_1^{M+1}}{1-(r_2/r_1)^{M+1}}u + \frac{r_1^{-M/2-1} - r_2^{M/2}/r_1^{M+1}}{1-(r_2/r_1)^{M+1}}d \to 0$$

as $M \to \infty$. Similarly,

$$\frac{\delta\gamma_{M/2-1}}{1-\delta} = \frac{r_1^{-M/2+1} - r_2^{M/2+2}/r_1^{M+1}}{1-(r_2/r_1)^{M+1}}u + \frac{r_1^{-M/2-2} - r_2^{M/2-1}/r_1^{M+1}}{1-(r_2/r_1)^{M+1}}d \to 0$$

as $M \to \infty$. Hence, there is $\overline{M}_E$ such that $\delta(V_{M/2} - V_{M/2-2}) = \delta(\gamma_{M/2} + \gamma_{M/2-1}) < (1-\delta)d$ for all $M \geq \overline{M}_E$ and even.

Analogously, if M is odd, we can show that both $\delta\gamma_{(M+1)/2}/(1-\delta)$ and $\delta\gamma_{(M-1)/2}/(1-\delta)$ converge to zero as M goes to infinity. Hence, there is $\overline{M}_O$ such that $\delta(V_{(M+1)/2} - V_{(M-3)/2}) < (1-\delta)d$ for all $M \geq \overline{M}_O$ and odd. Therefore, for all $M \geq \max\{\overline{M}_E, \overline{M}_O\}$, it follows that $\delta(\min_{m\in\{2,\dots,M\}} V_m^M - V_{m-2}^M) < (1-\delta)d$. ■

Recall that Proposition 3 states that the EM automaton I_M is a PEC if and only if $M \leq M^*$. Furthermore, M^* converges to infinity as δ converges to one.

Proof of Proposition 3. The second part of the proposition follows from Proposition 1.

Regarding the first part, note that if $M > M^*$, then either $\delta(V_1^M - V_0^M) \leq (1-\delta)d$ or $\delta(V_M^M - V_{M-2}^M) \leq (1-\delta)d$. Thus, it follows by Proposition 2 that I_M is not a PEC. Furthermore, Proposition 2 also implies that I_{M^*} is a PEC. Thus, to complete the proof it suffices to show that if I_{M+1} is a PEC, so is I_M. Clearly, this implies, by induction, that I_M is a PEC for all $M \leq M^*$.

For convenience, let $P(m, x)$ denote the system

$$\begin{pmatrix} 2 & -\delta & 0 & 0 & \cdots & 0 & 0 \\ -\delta & 2 & -\delta & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -\delta \\ 0 & 0 & 0 & 0 & \cdots & -\delta & 2 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{m-1} \\ \gamma_m \end{pmatrix} = \begin{pmatrix} (1-\delta)u \\ 0 \\ \vdots \\ 0 \\ x \end{pmatrix}. \tag{33}$$

So, in particular, $P(M, (1-\delta)d)$ is the same as (14). Let also $\bar{\gamma} = (\bar{\gamma}_1, \dots, \bar{\gamma}_{M+1})$ be the solution of $P(M+1, (1-\delta)d)$ and $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_M)$ be the solution of $P(M, (1-\delta)d)$.

Consider first the case where $\delta\bar{\gamma}_{M+1} \leq (1-\delta)d$. Then, $(\bar{\gamma}_1, \dots, \bar{\gamma}_M)$ solves $P(M, x)$ with $x = \delta\bar{\gamma}_{M+1} \leq (1-\delta)d$. Since $\hat{\gamma}$ solves $P(M, (1-\delta)d)$ and

$$\frac{\partial \gamma_m}{\partial x} = \frac{1-\delta}{\delta} \frac{r_1^i - r_2^i}{r_1^{M+1} - r_2^{M+1}} > 0,$$

it follows that $\hat{\gamma}_m \geq \bar{\gamma}_m$ for all $1 \leq m \leq M$. Hence, I_M is a PEC.

Suppose now that $\delta\bar{\gamma}_{M+1} > (1-\delta)d$. By Lemma 5, it follows that $\delta\bar{\gamma}_m > (1-\delta)d$ for all $1 \leq m \leq M+1$. We claim that $\delta\hat{\gamma}_M > (1-\delta)d$, which together with Lemma 5 establishes the proposition.

Suppose, in order to reach a contradiction, that $\delta\hat{\gamma}_M \leq (1-\delta)d$. This implies that $(r_1 - r_2)u \leq [(r_1 - 1)r_1^M + (1 - r_2)r_2^M]d$, which is equivalent to

$$\frac{u}{d} \leq \frac{r_1 - 1}{r_1 - r_2} r_1^M + \frac{1 - r_2}{r_1 - r_2} r_2^M := \overline{\theta}.$$

Since $\delta\bar{\gamma}_M > (1-\delta)d$, it follows that $(r_1^2 - r_2^2)u > [(r_1^2 - 1)r_1^M + (1 - r_2^2)r_2^M]d$, which is equivalent to

$$\frac{u}{d} > \frac{r_1^2 - 1}{r_1^2 - r_2^2} r_1^M + \frac{1 - r_2^2}{r_1^2 - r_2^2} r_2^M := \underline{\theta}.$$

We claim that $\underline{\theta} > \overline{\theta}$, which is a contradiction. Since $r_1^M > 1 > r_2^M$ and

$$\frac{r_1 - 1}{r_1 - r_2} + \frac{1 - r_2}{r_1 - r_2} = 1 = \frac{r_1^2 - 1}{r_1^2 - r_2^2} + \frac{1 - r_2^2}{r_1^2 - r_2^2},$$

the inequality $\underline{\theta} > \overline{\theta}$ is equivalent to

$$\frac{r_1^2 - 1}{r_1^2 - r_2^2} > \frac{r_1 - 1}{r_1 - r_2}. \tag{34}$$

Simplifying the latter inequality, one obtains

$$\frac{1-r_2}{r_1-r_2}r_1^2+\frac{r_1-1}{r_1-r_2}r_2^2>1,$$

which holds by Lemma 1. ■

4.3 Correlation under EM

We can reinterpret our model in terms of a dynamic risk sharing problem. To this end, assume that, in every period, each person receives an endowment of a single perishable and indivisible consumption good. The pair of endowments belongs to $\{(0,2),(2,0)\}$, that is, one person receives 2 units of the good while the other receives 0. As before, nature determines the endowments: if $\omega=1$ then $y=(y_\omega^1,y_\omega^2)=(2,0)$, while if $\omega=2$ then $y=(0,2)$.

We think of the player with an endowment of two units of the consumer good as the one that can provide the favor. The favor is interpreted as a transfer of one unit of the consumption good. We let $\tau_{\omega,t}^i$ denote the transfer made by individual i in state of nature ω in period t; it has to satisfy $0\leq\tau_{\omega,t}^i\leq y_\omega^i$ and $\tau_{\omega,t}^i\in\{0,1\}$. Once the decision regarding transfers is made, each individual consumes $c_{\omega,t}^i=y_\omega^i-\tau_{\omega,t}^i+\tau_{\omega,t}^{-i}$. The net transfer received by player 1 is $\theta_{\omega,t}=\tau_{\omega,t}^2-\tau_{\omega,t}^1$.

The EM automaton implies that the pattern of consumption and transfers satisfies the following properties.

Proposition 4 *For every equality matching automation I_M, there exists $\alpha>0$ such that*

1. $\mathrm{cov}(c_t^1,y_{t-k}^1|Y_{t-k})>0$ *and*
2. $\mathrm{cov}(\theta_t,\theta_{t-1}|Y_{t-1})<0$

for all $t\geq\alpha$ and $0\leq k\leq t-\alpha$.

It is easy to explain why this behavior leads to a positive correlation between current individual consumption and current and lagged income. If a consumer has a zero balance, she can consume if and only if she receives a positive endowment. Also, a consumer with a zero endowment today and a zero balance yesterday can consume today if and only if she received a positive endowment yesterday. Thus, the equality matching form of behavior can make current individual consumption and current and lagged individual income move together. A similar intuition holds for transfers.

Proof of Proposition 4. Since $Y_n(\omega) = 2$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$, we have that $\mathrm{cov}(c_t^1, y_{t-k}^1 | Y_{t-k}) = \mathrm{cov}(c_t^1, y_{t-k}^1)$ for all $0 \leq k < t$.

For convenience, let μ^n denote the uniform measure on Ω^n, i.e., $\mu^n(\omega^n) = 2^{-n}$ for all $\omega^n \in \Omega^n$.

By definition,

$$\mathrm{cov}(c_t^1, y_{t-k}^1) = \sum_{\omega^t \in \Omega^t} \frac{1}{2^t}(c_t^1(\omega^t) - \bar{c}_t^1)(y_{t-k}^1(\omega^t) - \bar{y}_{t-k}^1) \tag{35}$$

where $\bar{y}_{t-k}^1 = 1$. If $q_t(s)$ denotes the probability that in period t the state is $s \in \{0, \ldots, M\}$, then, $\bar{c}_t^1 = 1 + (q_t(M) - q_t(0))/2$. Since $\lim_{t\to\infty} q_t(s) = 1/(M+1)$ for all s, then $\bar{c}_t^1 \to 1$. Hence, we may compute $\mathrm{cov}(c_t^1, y_{t-k}^1)$ using $\bar{c}^1 = 1$ instead of $\bar{c}_t^1$, by considering t sufficiently large.

Let $\sigma_1(s)$ denote the probability that the state s_t is s and $\omega_{t-k} = 1$; similarly, let $\sigma_2(s)$ denote the probability that the state s_t is s when $\omega_{t-k} = 2$. We have that $\sigma_i(s) = \mu^t(\{\omega^t \in \Omega^t : \omega_{t-k} = i \text{ and } s_t(\omega^{t-1}) = s\})$ for $i = 1, 2$.

If $\omega_t = 1$, $s_t = M$ and $\omega_{t-k} = 1$ then $y_{t-k}^1 = 2$, $c_t^1 = 2$ and so

$$(c_t^1(\omega^t) - \bar{c}^1)(y_{t-k}^1(\omega^t) - \bar{y}_{t-k}^1) = 1.$$

Similarly, if $\omega_t = 1$, $s_t = M$ and $\omega_{t-k} = 2$ then $y_{t-k}^1 = 0$, $c_t^1 = 2$ and so

$$(c_t^1(\omega^t) - \bar{c}^1)(y_{t-k}^1(\omega^t) - \bar{y}_{t-k}^1) = -1.$$

Given $\omega_t = 1$, in all remaining cases we have

$$(c_t^1(\omega^t) - \bar{c}^1)(y_{t-k}^1(\omega^t) - \bar{y}_{t-k}^1) = 0,$$

since $c_t^1(\omega^t) = 1 = \bar{c}^1$.

For the case $\omega_t = 2$ we obtain

$$(c_t^1(\omega^t) - \bar{c}^1)(y_{t-k}^1(\omega^t) - \bar{y}_{t-k}^1) = \begin{cases} -1 & \text{if } s_t = 0 \text{ and } \omega_{t-k} = 1, \\ 1 & \text{if } s_t = 0 \text{ and } \omega_{t-k} = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

Then,

$$\operatorname{cov}(c_t^1, y_{t-k}^1) = \frac{\sigma_1(M) - \sigma_2(M)}{2} + \frac{\sigma_2(0) - \sigma_1(0)}{2}. \tag{37}$$

Thus, it is enough to show that $\sigma_1(M) \geq \sigma_2(M)$ and $\sigma_2(0) > \sigma_1(0)$.

For any $s \in S_M$ and $\omega^{k-2} = (\omega_{t-k+1}, \ldots, \omega_{t-1})$ let $\{s_j^i(s, \omega^{k-2})\}_{j=t-k+1}^t$ denote the sequence of states resulting from having state s in period $t-k$ and $\omega_{t-k} = i$ for $i = 1, 2$. Using the definition of T_M, one easily sees that $s_j^1(s, \omega^{k-2}) \geq s_j^2(s, \omega^{k-2})$ for any j, s and ω^{k-2}. So, given $s \in S_M$, if ω^{k-2} is such that $s_t^1(s, \omega^{k-2}) = 0$, then $s_t^2(s, \omega^{k-2}) = 0$. Similarly, if ω^{k-2} is such that $s_t^2(s, \omega^{k-2}) = M$, then $s_t^1(s, \omega^{k-2}) = M$. This implies that $\sigma_1(M) \geq \sigma_2(M)$ and $\sigma_2(0) \geq \sigma_1(0)$. Hence, it is enough to show that there exists $s \in S_M$, possible to reach at period $t-k$ starting from $\bar{s}_M$, for which the following holds: there exists ω^{k-2} such that $s_t^1(s, \omega^{k-2}) > 0$ and $s_t^2(s, \omega^{k-2}) = 0$ since this implies $\sigma_2(0) > \sigma_1(0)$.

Let $t \geq \alpha$ and $0 \leq k \leq t - \alpha$, i.e., $t - k \geq \alpha$. By choosing $\alpha > 0$ sufficiently large, any state $s \in S_M$ can be reached at period $t-k$ starting from $\bar{s}_M$: simply take $\omega = 2$ in the beginning in order to get to $s = 0$, then continue with $\omega = 2$ to keep $s = 0$ until period $t-k-s-1$, and take $\omega = 1$ from period $t-k-s$ until period $t-k-1$. If k is odd, let $s_{t-k} = 0$ and $\omega^{k-2} = (1, 2, 1, 2, \ldots)$. This will produce $s_t^1(s, \omega^{k-2}) = 1$ and $s_t^2(s, \omega^{k-2}) = 0$. If k is even, let $s_{t-k} = 1$ and $\omega^{k-2} = (2, 1, 2, 1, \ldots)$. Again, this will produce $s_t^1(s, \omega^{k-2}) = 1$ and $s_t^2(s, \omega^{k-2}) = 0$. This completes the proof that $\operatorname{cov}(c_t^1, y_{t-k}^1 | Y_{t-k}) > 0$.

Finally, we show that $\operatorname{cov}(\theta_t, \theta_{t-1}) < 0$ if t is sufficiently large.

Since $t_{\omega,n}^i \in \{0, 1\}$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$, then

$$\begin{aligned} \bar{\theta}_t &= \mu^t(\{\omega^t : s_t(\omega^t) > 0 \text{ and } \omega_t = 2\}) - \mu^t(\{\omega^t : s_t(\omega^t) < M \text{ and } \omega_t = 1\}) \\ &= \frac{1 - q_t(0) - 1 + q_t(M)}{2} = \frac{q_t(M) - q_t(0)}{2}. \end{aligned} \tag{38}$$

Letting $\bar{\theta} = 0$, it follows that $\bar{\theta}_t$ converges to $\bar{\theta}$ and so we can use $\bar{\theta}$ to compute $\text{cov}(\theta_t, \theta_{t-1})$ by considering t sufficiently large.

Note that $(\theta_t(\omega^t) - \bar{\theta})(\theta_{t-1}(\omega^t) - \bar{\theta}) = \theta_t(\omega^t)\theta_{t-1}(\omega^t)$ and that

$$\theta_t(\omega^t)\theta_{t-1}(\omega^t) = \begin{cases} 1 & \text{if } s_t(\omega^t) > 0, s_{t-1}(\omega^t) > 0, \omega_t = 2 \text{ and } \omega_{t-1} = 2, \\ 1 & \text{if } s_t(\omega^t) < M, s_{t-1}(\omega^t) < M, \omega_t = 1 \text{ and } \omega_{t-1} = 1, \\ -1 & \text{if } s_t(\omega^t) > 0, s_{t-1}(\omega^t) < M, \omega_t = 2 \text{ and } \omega_{t-1} = 1, \\ -1 & \text{if } s_t(\omega^t) < M, s_{t-1}(\omega^t) > 0, \omega_t = 1 \text{ and } \omega_{t-1} = 2. \end{cases} \tag{39}$$

Since, if $\omega_{t-1} = 2$, then both $s_t > 0$ and $s_{t-1} > 0$ if and only if $s_{t-1} > 1$, it follows that

$$\begin{aligned} \mu^t(s_t > 0, s_{t-1} > 0, \omega_t = 2, \omega_{t-1} = 2) &= \frac{\mu^t(s_t > 0, s_{t-1} > 0, \omega_{t-1} = 2)}{2} \\ &= \frac{\mu^t(s_{t-1} > 1, \omega_{t-1} = 2)}{2} = \frac{\sum_{s=2}^{M} q_t(s)}{4}. \end{aligned} \tag{40}$$

Similarly, if $\omega_{t-1} = 1$, then both $s_t < M$ and $s_{t-1} < M$ if and only if $s_{t-1} < M - 1$. Thus, it follows that

$$\begin{aligned} \mu^t(s_t < M, s_{t-1} < M, \omega_t = 1, \omega_{t-1} = 1) &= \frac{\mu^t(s_t < M, s_{t-1} < M, \omega_{t-1} = 1)}{2} \\ &= \frac{\mu^t(s_{t-1} < M - 1, \omega_{t-1} = 1)}{2} = \frac{\sum_{s=0}^{M-2} q_t(s)}{4}. \end{aligned} \tag{41}$$

If $\omega_{t-1} = 1$, then both $s_t > 0$ and $s_{t-1} < M$ if and only if $s_{t-1} < M$. Thus, it follows that

$$\begin{aligned} \mu^t(s_t > 0, s_{t-1} < M, \omega_t = 2, \omega_{t-1} = 1) &= \frac{\mu^t(s_t > 0, s_{t-1} < M, \omega_{t-1} = 1)}{2} \\ &= \frac{\mu^t(s_{t-1} < M, \omega_{t-1} = 1)}{2} = \frac{\sum_{s=0}^{M-1} q_t(s)}{4}. \end{aligned} \tag{42}$$

Finally, if $\omega_{t-1} = 2$, then both $s_t < M$ and $s_{t-1} > 0$ if and only if $s_{t-1} > 0$. Thus, it follows that

$$\begin{aligned} \mu^t(s_t < M, s_{t-1} > 0, \omega_t = 1, \omega_{t-1} = 2) &= \frac{\mu^t(s_t < M, s_{t-1} > 0, \omega_{t-1} = 2)}{2} \\ &= \frac{\mu^t(s_{t-1} > 0, \omega_{t-1} = 2)}{2} = \frac{\sum_{s=1}^{M} q_t(s)}{4}. \end{aligned} \tag{43}$$

It then follows that $\text{cov}(\theta_t, \theta_{t-1})$ converges to

$$-\lim_{t \to \infty} \frac{q_t(1) + q_t(M-1)}{4} = -\frac{1}{2(M+1)}. \tag{44}$$

Hence, if t is sufficiently large, we conclude that $\text{cov}(\theta_t, \theta_{t-1}) < 0$. ■

5 On the Optimality of Equality Matching

5.1 Symmetric Two-State Automata

An automaton is *symmetric* if (1) $S_1 = S_2$, $T_1 = T_2$, and $\bar{s}_1 = \bar{s}_2$; (2) there exists a bijection $\phi : S \to S$ such that $B_1(s) = B_2(\phi(s))$ and $\phi(T(1, s, a)) = T(2, \phi(s), a)$ for all $s \in S$ and $a \in A$.

Proposition 5 *If I is a symmetric two-state PEC, then I is the two-state EM automaton (i.e., $I = I_2$).*

Proof. Let $S_1 = S_2 = \{s_1, s_2\}$. Clearly, we cannot have $B_1(s_1) = B_1(s_2)$. In fact, if that common value for B_1 is P, then I is not SPE, while if it is NP, then each player could drop one state and obtain the same payoff. In both case, it would follows that I is not PEC. Therefore we may assume that $B_1(s_1) = P$ while $B_1(s_2) = NP$.

We claim that it cannot be that ϕ is the identity. Suppose that ϕ were the identity and let $s' = T(1, s_1, P)$. Since $\phi(s_1) = s_1$, it follows by symmetry that $T(2, s_1, P) = s'$. Since all transition must be used in along the play of I, then $T(1, s_1, NP) = T(2, s_1, NP) = s'$ as well. Hence, since s_1 is reached along the play of I, it follows that player 1 (and 2 as well) can profitably deviate from I. Indeed, by choosing NP his current gain is $(1 - \delta)d$ but the continuation payoff is the same since $T(1, s_1, P) = T(1, s_1, NP)$. This contradicts the fact that I is a PEC. Thus, $\phi(s_1) = s_2$ and $\phi(s_2) = s_1$, which implies $B_2(s_1) = NP$ and $B_2(s_2) = P$.

As before, it cannot be that $T(1, s_1, P) = T(2, s_1, NP)$ because then $T(1, s_1, NP) = T(1, s_1, P)$ (otherwise the transition $T(1, s_1, NP)$ would not be used during the play of I) and so player 1 could profit from a deviation from P to NP at state s_1.

To conclude, note that it suffices to show that $T(1, s_1, P) = s_2$. Indeed, this implies that $T(1, s_1, NP) = s_1$ (since I is SPE), and so $T(2, s_1, NP) = s_1$. By symmetry, $T(2, s_2, P) = s_1$, $T(2, s_2, NP) = s_2$ and $T(1, s_2, NP) = s_1$. Hence, I is the two-state EM automaton (recall that $T(2, s_1, P)$ can be either 0 or 1).

So, in order to reach a contradiction, suppose that $T(1, s_1, P) \neq s_2$, i.e., $T(1, s_1, P) = s_1$. Since I is SPE, then $T(1, s_1, NP) = s_2$ and so $T(2, s_1, NP) = s_2$ (since I

is PEC). By symmetry, it follows that $T(2, s_2, P) = s_2$, $T(2, s_2, NP) = s_1$ and $T(1, s_2, NP) = s_1$. These transitions imply that the discounted expected payoff of player 1 V_j at state $s = s_j$ satisfy:

$$V_1 = -\frac{(1-\delta)d}{2} + \delta\frac{V_1 + V_2}{2}$$
$$V_2 = \frac{(1-\delta)u}{2} + \delta\frac{V_1 + V_2}{2}.$$

Thus, $V_1 - V_2 = -(1-\delta)(u+d)/2$, and so player 1 can profitably deviate from P to NP at state s_1. This contradiction establishes the proposition. ■

5.2 Strong Symmetry and Long-Run Optimality of Equality Matching

An automaton is strongly symmetric if it is symmetric and the transition matrix of the Markov chain it induces is symmetric. An automaton is an asymptotic PEC if there exists $\delta^* \in (0, 1)$ such that it is a PEC for all $\delta \geq \delta^*$. For all $N \in \mathbb{N}$, let $\mathcal{A}_N$ be the set of all strongly symmetric, asymptotic PEC automata with a state space having no more than N elements.

Proposition 6 *For all $N \in \mathbb{N}$, every equality matching automaton I_M, with $M = N - 1$, solves*

$$\max_{I \in \mathcal{A}_N} \left[U_1(I) + U_2(I)\right].$$

Proposition 6 shows that the equality matching automaton is optimal (in the long-run) within the class of strongly symmetric asymptotic PEC.

Proof. Let I be a strongly symmetric asymptotic PEC. Note that for all $s \in S$, $I(s)$ is a strongly symmetric asymptotic PEC. Also, recall that S can be partitioned into several ergodic sets and possibly a transient set. The transient set has no impact on the long-run payoff and so it is enough to show that I_M yields an higher payoff than each ergodic set. Since the reduced matrix associated with each ergodic class is irreducible, it follows that it suffices to prove that I_M maximizes the sum of long-run payoffs within the class of strongly symmetric asymptotic PEC with irreducible transition matrices.

To that end, let I be a strongly symmetric asymptotic PEC with an irreducible transition matrix. It follows that the Markov chain on S has a unique stationary distribution (by irreducibility), which equals the uniform distribution (by strong symmetry). Hence,

$$U_i(I) = \frac{1}{2\,|S|}\sum_{s\in S}\sum_{\omega\in\Omega} u_i(\omega, B_\omega(s)).$$

Note that for all $i = 1, 2$, there exists $s \in S$ such that $B_i(s) = NP$. We establish this claim by contradiction. Suppose that for some $i \in \{1, 2\}$, we have $B_i(s) = P$, for all $s \in S$. Let $\tilde{I}_{-i}$ be such that $B_{-i}(s) = NP$, for all $s \in S$, which implies that $U_i(I_i, \tilde{I}_{-i}) = U^\delta_{-i}(I_i, \tilde{I}_{-i}) = u/2$ for all $\delta \in (0, 1)$. Suppose, in order to reach a contradiction, that $I_{-i} \neq \tilde{I}_{-i}$. Then, there is some state in which player $-i$ chooses P, and since I a strongly symmetric PEC with an irreducible transition matrix, it follows that

$$U_{-i}(I) \leq \frac{u}{2} - \frac{d}{2|S|} < U_{-i}(I_i, \tilde{I}_{-i}),$$

and so $U^\delta_{-i}(I) < U^\delta_{-i}(I_i, \tilde{I}_{-i})$ for all δ sufficiently close to 1. This is a contradiction since I is a subgame perfect equilibrium for all δ sufficiently close to 1. Thus, $I_{-i} = \tilde{I}_{-i}$, and so $U_i(I) = -d/2$. However, letting $\tilde{I}_i$ be such that $B_i(s) = NP$, for all $s \in S$, we obtain $U_i(\tilde{I}_i, I_{-i}) = U_i(\tilde{I}) = 0 > U_i(I)$ and so $U^\delta_i(\tilde{I}_i, I_{-i}) > U^\delta_i(I)$ for all δ close to 1, a contradiction.

Let $S_P = \{s \in S : B_2(s) = P\}$ and $S_{NP} = \{s \in S : B_2(s) = NP\}$. By symmetry, $|S_P| = |\{s \in S : B_1(s) = P\}|$ and $|S_{NP}| = |\{s \in S : B_1(s) = NP\}|$. Then, we obtain

$$\begin{aligned} U_1(I) =& \frac{1}{2\,|S|}\left(\sum_{s\in S} u_1(1, B(s)) + \sum_{s\in S} u_1(2, B(s))\right) \\ =& \frac{1}{2\,|S|}\left(-d\,|S_P| + u\,|S_P|\right) = \frac{|S_P|}{|S|}\frac{u-d}{2}. \end{aligned} \tag{45}$$

We have that $|S_{NP}| \geq 1$ and so $|S_P| = |S| - |S_{NP}| \leq |S| - 1$. Hence, it follows that,

$$U_1(I) \leq \frac{u-d}{2}\left(1 - \frac{1}{|S|}\right) \leq \frac{u-d}{2}\left(1 - \frac{1}{M+1}\right) = U_1(I_M). \tag{46}$$

Since, by symmetry, $U_2(I_M) = U_1(I_M) \geq U_1(I) = U_2(I)$, the result follows. ■

5.3 Example of Automata that are not PEC

The first example is obtained by changing the EM automaton so that there is forgiveness at the boundary. The state space and behavior function are the same as in the EM automaton I_M, the same being true for the transition for all $1 \leq m \leq M-1$

For $m = M$, let

$$T(\omega, m, a) = \begin{cases} M-1 & \text{if} \quad \omega = 2, a = P, \\ M & \text{if} \quad \omega = 2, a = NP, \\ M-1 & \text{if} \quad \omega = 1. \end{cases}$$

For $m = 0$, let

$$T(\omega, m, a) = \begin{cases} 1 & \text{if} \quad \omega = 1, a = P, \\ 0 & \text{if} \quad \omega = 1, a = NP, \\ 1 & \text{if} \quad \omega = 2. \end{cases}$$

This automaton is not a PEC since there are two transitions that are never used in the equilibrium path: those from $m = 1$ to $m = 1$ and from $m = M$ to $m = M$.

The second automaton is also obtained by modifying the EM automaton so that there is a higher return to gift-giving at the boundary. Consider the I_M automaton with $M = 4$ and change the following two transitions: $T(1, 0, P) = 2$ and $T(2, 4, P) = 2$. This automaton is not a PEC since player 2 can drop state 2.

More formally, let I_B denote the above automaton and define I_2 by $S_2 = \{0, 1, 3, 4\}$, $B_2(s) = NP$ if and only if $s = 0$,

$$\bar{s}_2 = \begin{cases} 3 & \text{if } \bar{s}_2^B = 2, \\ 4 & \text{if } \bar{s}_2^B = 3, \\ \bar{s}_2^B & \text{otherwise}, \end{cases} \tag{47}$$

$$T_2(1, s, a) = \begin{cases} 3 \text{ if } s = 0 \text{ and } a = P, \\ 3 \text{ if } s = 1 \text{ and } a = P, \\ 4 \text{ if } s = 3 \text{ and } a = P, \\ 3 \text{ if } s = 4 \text{ and } a = P, \\ 3 \text{ if } s = 4 \text{ and } a = NP, \end{cases} \tag{48}$$

$$T_2(2, s, a) = \begin{cases} 0 \text{ if } s = 0 \text{ and } a = NP, \\ 0 \text{ if } s = 1 \text{ and } a = P, \\ 1 \text{ if } s = 3 \text{ and } a = P, \\ 3 \text{ if } s = 4 \text{ and } a = P, \end{cases} \tag{49}$$

and define $T(\omega, s, a)$ arbitrarily in the remaining cases.

We claim that for all $\boldsymbol{\omega}$, the outcome $\mathbf{a}(I^B, \boldsymbol{\omega})$ induced by I^B is the same as the outcome $\mathbf{a}((I_1^B, I_2), \boldsymbol{\omega})$ induced by (I_1^B, I_2). In order to establish this claim, let Σ be the set of pairs of states $(s_1, s_2) \in S_1^B \times S_2$ with the property that there exists $\boldsymbol{\omega}$ and t such that $s_t(\boldsymbol{\omega}) = (s_1^t(\boldsymbol{\omega}), s_2^t(\boldsymbol{\omega})) = (s_1, s_2)$. Also, let $\Lambda = \{(0,0), (1,1), (2,3), (3,4), (4,4)\}$. It follows from the definition of I_1^B and I_2 that for all $\boldsymbol{\omega}$, $s_1(\boldsymbol{\omega}) = (\bar{s_1}, \bar{s_2}) \in \Lambda$ and that if $s_t(\boldsymbol{\omega}) \in \Lambda$, then $s_{t+1}(\boldsymbol{\omega}) \in \Lambda$ (see Table 2 below). Thus, $\Sigma \subseteq \Lambda$. Therefore, $\omega_t = 2$ and $\mathbf{a}_t(I^B, \boldsymbol{\omega}) = NP$ holds if and only if $s_1^t(\boldsymbol{\omega}) = 0$, which, due to $\Sigma \subseteq \Lambda$, holds if and only if $s_2^t(\boldsymbol{\omega}) = 0$. Since $\omega_t = 2$ and $\mathbf{a}_t((I_1^B, I_2), \boldsymbol{\omega}) = NP$ holds if and only if $s_2^t(\boldsymbol{\omega}) = 0$, it follows that player 2 fails to provide a favor under I^B exactly when he does it under (I_1^B, I_2). Hence, $\mathbf{a}(I^B, \boldsymbol{\omega}) = \mathbf{a}((I_1^B, I_2), \boldsymbol{\omega})$.

	$\omega_t = 1$	$\omega_t = 1$	$\omega_t = 2$	$\omega_t = 2$
s_t	a_t	s_{t+1}	a_t	s_{t+1}
(0,0)	P	(2,3)	NP	(0,0)
(1,1)	P	(2,3)	P	(0,0)
(2,3)	P	(3,4)	P	(1,1)
(3,4)	P	(4,4)	P	(2,3)
(4,4)	NP	(4,4)	P	(2,3)

Table 2: If $s_t \in \Lambda$, then $s_{t+1} \in \Lambda$

5.4 Symmetric Three-State Automata

The following tough EM automaton shows that not all symmetric PEC are EM.

The tough 3-state EM automaton is like the EM automaton with three states $(0, 1, 2)$ except that $B_2(1) = B_1(1) = NP$. The transitions are $T(1, 0, P) = 1$,

$T(1,1,NP) = 2$, $T(1,2,NP) = 2$, $T(2,2,P) = 1$, $T(2,1,NP) = 0$ and $T(2,0,NP) = 0$ on the equilibrium path and $T(1,0,NP) = 0$, $T(1,1,P) = 0$, $T(1,2,P) = 2$, $T(2,2,NP) = 2$, $T(2,1,P) = 2$ and $T(2,0,P) = 0$ outside.

Note that $T(2,1,NP) = 0$ (and similarly, $T(1,1,NP) = 2$), meaning that player 1 may lose credit even when player 2 does not provide the favor (one may interpret this as player 1 being forced to pay interest). Hence, this automaton is tougher than the EM, although it is just a variation of the EM. Therefore, like the asymmetric EM automaton, we still have asymptotic inefficiency and correlation between consumption and income. Furthermore, the tough EM automaton cannot maximize the sum of players' payoffs (this is clear if δ is very close to 1, since there is less production in this automaton compared with the three-state EM automaton).

The proof that no player can save neither on state nor on transitions is the same as for the EM automaton once we have shown that the tough EM automaton is a strict SPE. This is shown in the usual way. Player 1's value function is:

$$\begin{aligned}
v_0 &= -(1-\delta)\frac{d}{2} + \delta\frac{v_0 + v_1}{2} \\
v_1 &= -(1-\delta)\frac{d}{2} + \delta\frac{v_0 + v_2}{2} \\
v_2 &= (1-\delta)\frac{u-d}{2} + \delta\frac{v_1 + v_3}{2} \\
v_3 &= (1-\delta)\frac{u}{2} + \delta\frac{v_2 + v_4}{2} \\
v_4 &= (1-\delta)\frac{u}{2} + \delta\frac{v_3 + v_4}{2}.
\end{aligned}$$

Furthermore, define $\alpha_0 = (v_1 - v_0)/(1-\delta)$. Clearly, the tough EM is a strict SPE if $\delta\alpha_0 > d$.

Solving, one obtains

$$\begin{aligned}
a_0 &= \frac{u\delta + 2d}{4-\delta^2}, \\
v_0 &= \frac{u\delta^2 - 4d + 2d\delta + d\delta^2}{8-2\delta^2}, \\
v_1 &= \frac{\delta(u-d)}{4+2\delta} \\
v_2 &= \frac{4u - 2u\delta - u\delta^2 - d\delta^2}{8-2\delta^2}
\end{aligned}$$

By computing the limit as δ goes to one, it follows that $\delta\alpha_0 > d$ if δ is sufficiently large.

We can also establish that the tough EM automaton does not maximize the sum of players' payoffs. Letting V_s denote player 1's discounted expected payoff in state s under the three-state EM automaton, we have

$$\begin{aligned}
V_0 &= \frac{u\delta - 2d + d\delta^2}{4-\delta^2}, \\
V_1 &= \frac{u-d}{2+\delta}, \\
V_2 &= \frac{2u - u\delta^2 - d\delta}{4-\delta^2}
\end{aligned}$$

Note that in both automata player 2's payoff in state s is equal to player 1's payoff in state $2-s$. So, if the initial state is s, the sum of players' payoffs is $U_s = V_s + V_{2-s}$ under the three-state EM automaton and $u_s = v_s + v_{2-s}$ under the tough EM automaton. Solving, we obtain

$$\begin{aligned}
U_0 - u_0 &= \frac{\delta(u-d)}{2+\delta} > 0 \\
U_1 - u_1 &= \frac{(2-\delta)(u-d)}{2+\delta} > 0
\end{aligned}$$

and $U_2 - u_2 = U_0 - u_0 > 0$.

A Appendix

A.1 Proof of Equation 10

From the first equality in (9), we obtain that

$$\begin{aligned}
&2V_M - \delta V_M = (1-\delta)u + \delta V_{M-1} \Leftrightarrow \\
&V_M = \frac{1-\delta}{2-\delta}u + \frac{\delta}{2-\delta}V_{M-1} + V_{M-1} - V_{M-1} \Leftrightarrow \\
&V_M - V_{M-1} = \frac{1-\delta}{2-\delta}u - \frac{2(1-\delta)}{2-\delta}V_{M-1} \Leftrightarrow \\
&\frac{V_M - V_{M-1}}{1-\delta} = \frac{u}{2-\delta} - \frac{2V_{M-1}}{2-\delta}.
\end{aligned}$$

From the second equality in (9), we obtain that

$$
\begin{aligned}
&2V_m = (1-\delta)(u-d) + \delta(V_{m+1} - V_m) + \delta(V_m + V_{m-1}) \Leftrightarrow \\
&2V_m - \delta V_m = (1-\delta)(u-d) + \delta(V_{m+1} - V_m) + \delta V_{m-1} \Leftrightarrow \\
&V_M = \frac{1-\delta}{2-\delta}(u-d) + \frac{\delta}{2-\delta}(V_{m+1} - V_m) + \frac{\delta}{2-\delta}V_{m-1} + V_{m-1} - V_{m-1} \Leftrightarrow \\
&V_m - V_{m-1} = \frac{1-\delta}{2-\delta}(u-d) + \frac{\delta}{2-\delta}(V_{m+1} - V_m) - \frac{2(1-\delta)}{2-\delta}V_{m-1} \Leftrightarrow \\
&\frac{V_m - V_{m-1}}{1-\delta} = \frac{u-d}{2-\delta} + \frac{\delta}{2-\delta}\frac{V_{m+1} - V_m}{1-\delta} - \frac{2V_{m-1}}{2-\delta}.
\end{aligned}
$$

A.2 Examples concerning extensions of Lemma 5

Consider the following statement: *If* $\delta(V_M - V_{M-2}) \geq (1-\delta)d$*, then* $V_m - V_{m-1} > V_{m+1} - V_m$ *for all* $1 \leq m \leq M-1$.

Recall that $\gamma_m = V_m - V_{m-1}$, and so the conclusion can be written as $\gamma_m > \gamma_{m+1}$.

This proposition is false: When $M = 4$, $u = 1.5$, $d = 1$ and $\delta = 0.81$, we obtain:

$\delta(V_4 - V_2)/(1-\delta)$	$\delta(\gamma_3 - \gamma_4)/(1-\delta)$	$\delta[(V_3 - V_1) - (V_4 - V_2)]/(1-\delta)$
1.026986	-0.1416	-0.075509577

This example also shows that the following proposition is false: *If* $\delta(V_M - V_{M-2}) \geq (1-\delta)d$*, then* $V_m - V_{m-2} > V_{m+1} - V_{m-1}$ *for all* $2 \leq m \leq M-1$.

The following propositions are also false: (1) *If* $\delta(V_M - V_{M-2}) \geq (1-\delta)d$*, then* $\delta(V_m - V_{m-1}) > (1-\delta)d$ *for all* $1 \leq m \leq M$. (2) *If* $\delta(V_M - V_{M-2}) \geq (1-\delta)d$*, then* $\delta(V_m - V_{m-2}) > (1-\delta)d$ *for all* $2 \leq m \leq M$.

The example given above shows this (recall $d = 1$):

$\delta(V_4 - V_2)/(1-\delta)$	$\delta(V_4 - V_3)/(1-\delta)$	$\delta(V_3 - V_1)/(1-\delta)$
1.026986	0.584291	0.951476

References

DOW, M. (2003): "Explicit Inverses of Toeplitz and Associated Matrices," *ANZIAM Journal*, 44, E215–E215.

Fafchamps, M., and S. Lund (2001): "Risk-Sharing Networks in Rural Philippines," mimeo, University of Oxford.

——— (2003): "Risk-Sharing Networks in Rural Philippines," *Journal of Development Economics*, 71, 261–287.

Fiske, A. (1992): "The Four Elementary Forms of Sociality: Framework for a Unified Theory of Social Relations," *Psychological Review*, 99, 689–723.

Kocherlakota, N. (1996): "Implications of Efficient Risk Sharing without Commitment," *Review of Economic Studies*, 63, 595–609.

Koeppl, T. (2006): "Differentiability of the Efficient Frontier when Commitment to Risk Sharing is Limited," *Topics in Macroeconomics*, 6, Article 10.

La Ferrara, E. (2003): "Kin Groups and Reciprocity: A Model of Credit Transactions in Ghana," *American Economic Review*, 93, 1730–1751.

Rincón-Zapatero, J., and M. Santos (2006): "Differentiability of the Value Function without Interiority Assumptions," mimeo, Universidad Carlos III de Madrid and Miami University.

Rudin, W. (1976): *Principles of Mathematical Analysis.* McGraw-Hill, New York, 3rd edn.

Townsend, R. (1994): "Risk and Insurance in Village India," *Econometrica*, 62, 539–591.