

# Social Interactions and Spillovers: Incentives, Segregation and Topology\*

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## Abstract

The aim of this paper is to understand the interactions between productive effort and the creation of synergies that are the sources of technological collaboration agreements, agglomeration, and social interactions. We model this interaction in a way that allows us to characterize how agents devote resources to both activities. This permits a full-fledged equilibrium/welfare analysis of network formation with endogenous productive efforts, to derive unambiguous comparative statics results and to analyze community sorting by individual traits. In spite of its parsimony the model retains enough richness to replicate a broad range of empirical regularities displayed by social and economic networks, and to relate them to individual and social welfare.

**Key words:** random network, spillovers, network formation, network topology.

**JEL Classification:** L22, L51, O31, O38

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# 1 Introduction

External effects (spillovers) pervade economies and societies in general. Both inter- and intra-industry cross interaction between firms have been the object of studies, since at least the work of Marshall (1890), for the former, and Jacobs (1969), for the latter.<sup>1</sup> Social interactions also have a crucial importance in the determination of individuals' well-being, as pointed early by Becker (1974) and recently emphasized by the literature on social capital.<sup>2</sup>

Given the importance and pervasiveness of these external effects, it is natural that individuals and firms may want to control and manipulate the size and scope of those external benefits to their advantage. For example, regional economists have convincingly shown that economic agents agglomerate in few locations in the economic landscape, precisely in order to reap these localization externalities (Ciccone and Hall 1996, Guiso and Schivardi 2007).<sup>3</sup> In a similar vein, it is difficult to understand technological collaboration agreement between firms ("joint ventures" and other similar contracts) without thinking that these are done to control external effects (d'Aspremont and Jacquemin 1988).<sup>4</sup> Finally, the persistent stratification of social groups among many dimensions (such as income, race, education) is *prima facie* evidence of the desire of social groups to arrange themselves so as to internalize spillovers (Tiebout 1956, Benabou 1993).

In this paper, we investigate the interaction between productive effort and the creation of synergies. We model this in a way that allows us to characterize how agents devote resources to both activities optimally. In turn, this permits a full-fledged equilibrium/welfare analysis of individual decisions and to derive unambiguous comparative statics results. In particular, optimal individual decisions induce preferences over the composition of the interacting group, and we analyze segregating forces deriving from these induced preferences. The model is also flexible enough to be brought to the data.

**The environment** Our model has three main ingredients.

First, we consider a model with local complementarities in productive investment. More precisely, spillovers are generated by paired agents and are multiplicative in own's and other's productive effort. We allow for two different sources of heterogeneity. On the one hand, agents can differ in their marginal returns to own productive effort. On the other hand, for identical levels of productive efforts, spillovers can vary with the strength of the synergistic linkage across different pairs of agents. It turns out that this payoff structure allows to pin down exactly how the level of productive effort varies with the pattern of external effects exerted on each individual, and with the idiosyncratic characteristics of the agents.

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<sup>1</sup>See Duranton and Puga (2004) for a review of this literature.

<sup>2</sup>Coleman (1990) and Putnam (2000) are standard references. Sobel (2002) and Durlauf (2002) offer critical surveys of this literature.

<sup>3</sup>See Rosenthal and Strange (2004) for a survey.

<sup>4</sup>See also Suzumura (1992).

Second, we assume that agents devote a (joint) amount of resources to building synergies with others. The collection of socialization efforts determines the meeting possibilities across pairs of agents and results in a pattern of, possibly heterogeneous, bilateral interactions. The intensity of each bilateral interaction which results from this socialization process becomes a scaling factor on the local production complementarities arising between these two agents.

Third, while socialization is not earmarked in that individuals do not invest in nominal social relationships, it is anyway circumscribed to the boundaries of the community to which every agent belongs. Given that individuals have idiosyncratic traits, a community is characterized by a population size and a distribution of individual characteristics.

An innovation of our study is, precisely, that the synergistic effort is generic within a community –a scalar decision. Socializing is not equivalent, in our approach, to elaborating a nominal list of intended relationships, as in the literature on network formation surveyed by Jackson (2005). This is realistic in many applications,<sup>5</sup> particularly when networks are so large that keeping track of every participant becomes a burdensome task. In addition, this shortcut greatly improves the tractability of the model. Unlike with richer models of link formation, we can resort to off-the-shelf Nash equilibrium analysis without being burdened by the extreme (combinatorial) multiplicity problems of the other models.<sup>6</sup> As a result, we can perform a standard type of equilibrium analysis that equates marginal costs and benefits of both production and socialization. Of course, this equilibrium characterization also greatly simplifies welfare and comparative statics analyses. At the same time, our analysis permits to draw a close connection between topological features and welfare and economic implications.

We believe that this link between topology, equilibrium, community composition and welfare is the main value added of the model.

**Results** We first characterize the equilibria of the model, when agents take their decision about their productive effort and their socialization effort simultaneously. We show that there are two interior equilibria and one (partially) corner equilibrium, when a sufficiently large number of individuals is implicated. The (partially) corner equilibrium where agents do not invest at all in

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<sup>5</sup>Researchers go to fairs, or congresses to listen, to be listened to, and to meet other investigators in general. More generally, face-to-face meetings among agents that share a common location often result from random encounters among these agents, as the early literature on segregation indexes already points out (Bell 1954).

<sup>6</sup>In a typical game of network formation, players simultaneously announce all the links they wish to form with. The links that form are those that are mutually announced by both partners. The cost of creating and maintaining links are then paid. As a consequence of the large multi-dimensional strategy space, and because link creation requires the mutual consent of the two involved parties, a severe coordination problem arises. As such, the game often displays a multiplicity of Nash equilibria, and very different network geometries can arise endogenously. A partial solution to this problem can be found by allowing pair-wise or coalitional deviations, or by restricting to cooperative-like network stability notions (Jackson and Wolinsky, 1996). Jackson (2005) surveys this literature, while Calvó-Armengol and Ilkiliç (2006) derive the connections between this approach and standard game-theoretic refinements.

building synergies is unstable. Instead, the two interior equilibria are stable. Existence and stability of interior equilibria are obtained when the level of cross synergies as well as the heterogeneity in individual traits are not too high, which amounts to bounding from above a compound index of both payoff parameters.

For large enough populations, equilibrium actions take a particularly simple form. Recall that agents can display different marginal returns to own productive effort. We label “individual type” the value of this marginal return at the origin. We first show that the ratios of productive as well as socialization efforts across different pairs of agents are all equal to the ratio of their individual types. In other words, at equilibrium, the productive and socialization efforts for a given agent are the product of his individual type with some baseline values for the productive and socialization efforts. These baseline values, in turn, are obtained from a system of two equations with two unknowns that admits exactly two positive solutions –hence the two interior equilibria. Compared to the case of an isolated agents, these baseline values are all scaled up, by a *synergistic multiplier* which homogeneous across players and which depends on a *compound index of heterogeneity* in the community within which they interact.

This simple equilibrium characterization has a number of interesting implications. In particular, we can show that one of the interior equilibria displays both higher socialization and productive effort than the other, so that we can talk of high-action and low-action equilibrium. It also turns out that the high-actions equilibrium is Pareto superior. Furthermore, the socially efficient outcome lies in between the two equilibria, so that we can effectively talk about a too-high and a too-low equilibrium.

An important question is then how an exogenous change in the returns to production and socialization affect the relative production and socialization efforts at equilibrium. It turns out that, when the returns increase, all equilibrium actions decrease at the Pareto-superior equilibrium, while they increase at the Pareto-inferior equilibrium. In both cases, the percentage change in socialization effort is higher (in absolute value) than that of the productive effort. We think this may provide an explanation, for example, of the large increase in agreements of collaboration in R&D in the recent past (Caloghirou, Ioannides, Vonortas 2003). It could also explain the decline in social capital documented by Putnam (2000).

The equilibrium payoffs have sorting-inducing effects. More precisely equilibrium payoffs depend on the *synergistic multiplier* which itself depends on the *compound index of heterogeneity*. Therefore changes in the composition of the population have a direct impact on equilibrium payoffs, asymmetric across different types. This asymmetry provides incentives for agents at the two extremes of the type distribution to segregate themselves from the rest, as long as segregating institutions or mechanisms are available. This segregation, when it occurs yields a more efficient community assignment than the equilibrium with all the population mixed within the same community.

We then turn to the implications of the model for the topology of networks. When synergy values are all between zero and one, our equilibrium socialization efforts can be interpreted as a (multinomial) random graph with independent link probabilities, where the expected number of links accruing to each agent (also known as the degree of the corresponding network node), is equal to the socialization effort of this agent. Models of random graph with given expected degree sequence (here, the equilibrium profile of socialization efforts) have been analyzed by Chung and Lu (2002). They can replicate many of the observed features of real-life networks. We examine the implication of our model for the connectivity distribution, the clustering (namely, the propensity of friends of an individual to be friends between themselves), and the average distance, as well as the emergence of a giant component, that is, the existence of a connected sub-network encompassing a large fraction of the population.

To summarize, we propose a methodology that can usefully relate network topology to economic features of the model (and vice versa), which is an advantage with respect to other models that replicate well observed network topology.<sup>7</sup>

The paper is organized as follows. Section 2 describes the model, and introduces the baseline game as well as the replica game. Section 3 contains the equilibrium and welfare analysis. The comparative statics results are gathered in Section 4. Section 5 discuss the equilibrium and welfare implications for (and from) the network topology. Section 6 analysis the particular case of homogeneous populations with general cost structures, and the emergence of giant components. All proofs are gathered in the appendix.

## 2 The game

**The replica game**  $N = \{1, \dots, n\}$  is a finite set of players, and  $T = \{1, \dots, t\}$  is a finite set of types for these players. We let  $n$  be a multiple of  $t$ , that is,  $n = mt$  for some integer  $m \geq 1$ , so that there is the same number of players of each type.

More precisely, we refer to the case  $n = t$  as *the baseline game*, and to the general case  $n = mt$  as *the  $m$ -replica* of this baseline game. In an  $m$ -replica game, there are exactly  $m$  players of each type  $\tau \in T$ .

For each player  $i \in N$ , we denote by  $\tau(i) \in T$  his type.

We consider a simultaneous move game of network formation and investment. The returns to the investment are the sum of a private component and a synergistic component. The private returns are heterogeneous across players and depend on their type. We denote by  $\mathbf{b} = (b_1, \dots, b_t)$  the profile of these private returns, where  $0 < b_1 \leq b_2 \leq \dots \leq b_t$ . Even though each type in the replica game has the same number of individuals, we can match any finite distribution of types in

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<sup>7</sup>Kirman (1983), Kirman, Oddou and Weber (1986) and Ioannides (1990) propose and analyze early models relating random networks of interaction with economic outcomes.

a population by adding multiple copies of an individual type.<sup>8</sup>

The synergistic returns depend on the network formed on account of individual choices, as described below.

**Network formation** Consider some  $m$ -replica game,  $m \geq 1$ . Let  $n = mt$ .

Each player  $i$  selects a socialization effort,  $s_i \geq 0$ . Let  $\mathbf{s} = (s_1, \dots, s_n)$  be a profile of socialization efforts. Then,  $i$  and  $j$  interact with a link intensity given by:

$$g_{ij}(\mathbf{s}) = \rho(\mathbf{s}) s_i s_j \quad (1)$$

By definition, links are symmetric, that is,  $g_{ij} = g_{ji}$ . We also allow for self-loops (when  $i = j$ ). The total interaction intensity for a player  $i$  is:

$$g_i(\mathbf{s}) = \sum_{j=1}^n g_{ij}(\mathbf{s}) = \rho(\mathbf{s}) s_i \sum_{j=1}^n s_j.$$

We set

$$\rho(\mathbf{s}) = \begin{cases} 1 / \sum_{j=1}^n s_j, & \text{if } \mathbf{s} \neq \mathbf{0} \\ 0, & \text{if } \mathbf{s} = \mathbf{0} \end{cases} \quad (2)$$

so that  $g_i(\mathbf{s}) = s_i$ . That is, players decide upon their total interaction intensity.

In this model, the exact identity of the interacting partner is not an object of choice. Rather, players choose an aggregate level of socialization effort. This total effort is then distributed across each and every possible bilateral interaction in proportion to the partner's socialization effort. This interaction pattern arises naturally when meetings result from casual encounters rather than from an ear-marked socialization process.

The functional form in (1) and (2) can be tied back to simple properties of the link intensity  $g_{ij}(\mathbf{s})$ , as established below.

**LEMMA 1** *Suppose that, for all  $\mathbf{s} \neq \mathbf{0}$ , the link intensity satisfies:*

(A1) *symmetry:  $g_{ij}(\mathbf{s}) = g_{ji}(\mathbf{s})$ , for all  $i, j$ ;*

(A2) *aggregate constant returns to scale:  $\sum_{j=1}^n g_{ij}(\mathbf{s}) = s_i$ ;*

(A3) *anonymous socialization:  $g_{ji}(\mathbf{s}) / s_j = g_{ki}(\mathbf{s}) / s_k$ , for all  $i, j, k$ ;*

*then, the link intensity is given by (1) and (2).*

Conditions (A2) and (A3) reflect the fact that  $i$  controls his total number of contacts  $s_i$ , but the actual type composition of these contacts depends on others' socialization efforts. More precisely, each player devotes the same share of his total socialization effort to interacting with player  $i$ .

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<sup>8</sup>With enough replications, of course.

When  $\max_i s_i^2 < 1/\rho(\mathbf{s})$ , all link intensities are between 0 and 1. In this case, we can view the network as a random graph where  $g_{ij}(s)$  is the probability of having an edge between  $i$  and  $j$ , and links are independent across different pairs of players. This random graph model with given expected degrees  $\mathbf{s} = (s_1, \dots, s_n)$  is analysed, e.g., in Chung and Lu (2002). This model can replicate many of the connectivity distributions encountered in real-life networks, and displays some other interesting topological features that we analyse in Section 5.<sup>9</sup>

**Investment** Each player  $i$  makes a productive investment,  $k_i \geq 0$ . The choices of  $s_i$  and  $k_i$  are simultaneous. Let  $\mathbf{k} = (k_1, \dots, k_n)$  be a profile of investments. These individual investments yield both a private and a synergistic return.

The private return to player  $i$  depend only on his own investment level  $k_i$  and his idiosyncratic traits, summarized by  $b_{\tau(i)} \geq 0$ . We adopt a simple quadratic expression  $b_{\tau(i)} k_i - k_i^2/2$ .

The synergistic return depends on both  $\mathbf{s}$  and  $\mathbf{k}$ . We consider multiplicative synergies  $k_i k_j$  that are additively separable across pairs of players. For each pair of players  $i, j$ , these cross effects are scaled by a factor that reflects the intensity of the interaction between the two players. More precisely, we assume that:

$$\frac{\partial^2 u_i(\mathbf{s}, \mathbf{k})}{\partial k_i \partial k_j} = a g_{ij}(\mathbf{s}), \text{ for all } i \neq j, \quad (3)$$

where  $a \geq 0$  corresponds to the level of synergistic returns.

Notice that the symmetry (A1) in Lemma 1 is tantamount to payoffs being twice continuously differentiable in the productive investments  $\mathbf{k}$ .

**Payoffs** Let  $c > 0$ . Player  $i$ 's utility is given by:

$$u_i(\mathbf{s}, \mathbf{k}) = b_{\tau(i)} k_i + a \sum_{j=1, j \neq i}^n g_{ij}(\mathbf{s}) k_j k_i - \frac{1}{2} c k_i^2 - \frac{1}{2} s_i^2 \quad (4)$$

Payoffs are a linear-quadratic function of  $k_i$ s with non-negative cross effects (3) reflecting strategic complementarities in productive investments. The size  $a g_{ij}(\mathbf{s}) \geq 0$  of these complementarities depends on the profile of socialization efforts, and varies across different pairs of players.

Payoffs also display strategic complementarities in socialization efforts,  $\partial^2 u_i(\mathbf{s}, \mathbf{k}) / \partial s_i \partial s_j \geq 0$ .

### 3 Equilibrium analysis and Pareto ranking of equilibria

#### 3.1 Equilibrium analysis

We solve for the interior Nash equilibria in pure strategies  $(\mathbf{s}^*, \mathbf{k}^*) = (s_1^*, \dots, s_n^*, k_1^*, \dots, k_n^*)$  of the  $m$ -replica game with heterogeneous types  $\mathbf{b} = (b_1, \dots, b_t)$ , and for  $m$  large enough.

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<sup>9</sup>See Ioannides (2006), Jackson (2007) and Vega-Redondo (2007) for more on random graphs and their connections to economics.

Under some conditions that we provide, there are exactly three such equilibria.

In one (partially corner) equilibrium, the level of socialization effort is null for all players. Indeed, non-reciprocated unilateral socialization efforts do not yield to any interaction supportive of synergies.

The two other equilibria are interior.

We first identify the (partially) corner equilibrium of the game.

**LEMMA 2** *For all  $m$ -replica game,  $(s_i^*, k_i^*) = (0, b_{\tau(i)}/c)$  for all  $i = 1, \dots, mt$  is a pure strategy Nash equilibrium with corresponding equilibrium payoffs  $b_{\tau(i)}^2/2c$ . For  $m$  high enough, this equilibrium is not stable for the best-reply dynamics.*

This is a strict equilibrium, thus it cannot be discarded on the basis of standard refinements. However, this equilibrium is not stable for sufficiently large populations. Indeed, any socialization effort, however small, leads to a social interaction that generates synergy payoffs when at least one additional agent socializes as well, which induces every other agent to socialize further.

Given that the partially corner equilibrium is un-stable for high enough replications, we concentrate on the interior equilibria.

Define:

$$a(\mathbf{b}) = a \frac{\sum_{\tau=1}^t b_{\tau}^2}{\sum_{\tau=1}^t b_{\tau}}. \quad (5)$$

Holding the average type  $\sum_{\tau=1}^t b_{\tau}/t$  constant, the parameter  $a(\mathbf{b})$  increases with the heterogeneity in types. More generally,  $a(\mathbf{b})$  increases with the ratio  $\sum_{\tau=1}^t b_{\tau}^2 / \sum_{\tau=1}^t b_{\tau}$ , which many authors refer to as *the second-order average type* (e.g., Vega-Redondo 2007). When types are all homogeneous, that is,  $b_1 = \dots = b_t = b$ , we have  $a(\mathbf{b}) = ab$ .

**THEOREM 1** *Suppose that  $2(c/3)^{3/2} > a(\mathbf{b}) > 0$ . Then, there exists an  $m^*$  such that for all  $m$ -replica games with  $m \geq m^*$ , there are exactly two interior pure strategy Nash equilibria. These pure strategy Nash equilibria are such that, for all players  $i$  of type  $\tau$ , the strategies  $(s_i, k_i)$  converge to  $(s_{\tau(i)}^*, k_{\tau(i)}^*)$  as  $m$  goes to infinity, where  $s_{\tau(i)}^* = b_{\tau(i)}s$ ,  $k_{\tau(i)}^* = b_{\tau(i)}k$ , and  $(s, k)$  are positive solutions to:*

$$\begin{cases} s = a(\mathbf{b})k^2 \\ k [c - a(\mathbf{b})s] = 1 \end{cases}. \quad (6)$$

*For  $m$  high enough, these two Nash equilibria are stable for the best-reply dynamics.*

When  $a(\mathbf{b})$  is small enough compared to the infra-marginal cost for a productive investment, the system of two equations (6) with two unknowns has exactly two positive solutions. As  $m$  gets large, each such solution gets arbitrarily close to a pure strategy Nash equilibrium of the corresponding  $m$ -replica game. We get two approximate Nash equilibria. Besides, as  $m$  gets large, every pure strategy Nash equilibrium gets arbitrarily close to a solution of (6).

Table 1 shows the discrepancy between equilibrium and approximated equilibrium actions for various population sizes. The last column corresponds to the approximated equilibrium actions; the other columns give the exact Nash equilibrium actions as  $m$  varies.

Table 1: Simulations with  $a = 2, c = 1, t = 1$  and  $b_1 = 0.1$ .<sup>10</sup>

$m$	2	5	10	20	50	100	500	$\infty$
Low equilibrium								
$k^*$	1,898	1,195	1,101	1,065	1,049	1,046	1,046	1,046
$s^*$	2,366	815	458	303	234	222	218	219
High equilibrium								
$k^*$	3,346	4,643	4,591	4,508	4,444	4,420	4,400	4,394
$s^*$	3,506	3,923	3,911	3,891	3,875	3,869	3,864	3,862

For an homogeneous population with common trait  $b$ , one can verify that the exact equilibrium equations are:

$$\begin{cases} s = abk^2 \left(1 - \frac{1}{m}\right)^2 \\ k \left[ c - abs \left(1 - \frac{1}{m}\right) \right] = 1 \end{cases}$$

Comparing with (6), it can be checked that the approximation error is of the order of  $m^{-3/2}$ . In particular, when  $m = 100$ , the approximation error is of the order  $10^{-3}$ .

The approximated equilibria  $(\mathbf{s}^*, \mathbf{k}^*)$  characterized in Theorem 1 display three important features.

First, the level of socialization per unit of productive investment is the same for all players, that is,  $s_i^*/k_i^* = s_j^*/k_j^*$ , for all  $i, j$ . Given our quadratic cost structure, this is equivalent to having a marginal rate of substitution of socialization versus investment for gross benefits uniform across all players. An approximate equilibrium is thus fully characterized by specifying the value of this ratio together with the population profile of productive investments.

Second, differences in productive investments reflect differences in idiosyncratic traits. More precisely,  $k_i^*/k_j^* = b_{\tau(i)}/b_{\tau(j)}$ , for all  $i, j$ . Absent any synergy payoffs, private returns are maximal when productive investment is equal to  $b_{\tau(i)}/c$ , and the ratio of these maximal investments is then also equal to  $b_{\tau(i)}/b_{\tau(j)}$ . At an approximate equilibrium, the relative value of productive investments thus remains unchanged with and without synergies. The presence of synergies only affects the absolute value of those investments.

Third, in the presence of synergies, productive investments are all scaled up (compared to the case without synergies) by a *synergistic multiplier* given by:<sup>11</sup>

$$\frac{1}{1 - a(\mathbf{b})s^*/c}. \quad (7)$$

<sup>10</sup>Numbers are multiplied by  $10^4$ .

<sup>11</sup>This is obtained by rewriting the second equation of (6) as  $k_i^* = b_{\tau(i)}k^* = b_{\tau(i)}/(c - a(\mathbf{b})s^*)$ , and by dividing the right-hand term by the optimal productive investment level in the absence of synergies,  $b_{\tau(i)}/c$ .

This synergistic multiplier, which is homogeneous across all players, is a decreasing function of the infra-marginal productive investment cost  $c$ , and an increasing function of the second-order average type  $a(\mathbf{b})$ . Beyond these dependence on exogenous payoff parameters, the synergistic multiplier also depends on the endogenous baseline equilibrium socialization effort  $s^*$ .

In substance, individual traits enter multiplicatively into own actions at approximate equilibria. Also, the population-wide heterogeneity collapses into a single index,  $a(\mathbf{b})$ . Compared on the case without synergies, productive investments are all scaled up by a common synergistic multiplier that depends on this index. Finally, the ratio of socialization to productive effort is uniform across all agents.

These properties of the equilibrium actions have implications for the type composition of the social interaction circle of each player. Recall that  $s_i^*$  gives the total interaction intensity, or size, of the social circle of agent  $i$ . We deduce from the above discussion that social circles across players of different types  $\tau$  and  $\tau'$  vary in fixed proportions  $b_\tau/b_{\tau'}$ . Beyond this size effect, the inner composition of social circles is the same for all agents. More precisely, every agent  $i$  devotes an identical fraction  $b_{\tau'}/\sum_{\tau \in T} b_\tau$  of his total socialization intensity to interacting with type  $\tau'$  agents, and this fraction is independent of his own type.

Figure 1 plots equations (6).

From the graph, it is clear that the system (6) needs not always have a non-negative solution. The upper bound on  $a(\mathbf{b})$  in Theorem 1 is a necessary and sufficient for the two graphs to cross in the positive orthant of the space  $(s, k)$ . When  $a(\mathbf{b})$  is too large, the synergistic multiplier operates too intensively and there is no intersection. More precisely, holding  $s$  fixed, the level of productive investments escalates without bound. In turn, this triggers and unbounded increase in socialization effort, and both effects positively feed back into each other.

**REMARK 1** *When  $0 < a(\mathbf{b}) < 2(c/3)^{3/2}$ , the system of equations (6) has two different non-negative solutions. When  $a(\mathbf{b}) = 2(c/3)^{3/2}$ , there is a unique non-negative solution  $(s, k) = (3/c, \sqrt{3/c})/2$ . When  $a(\mathbf{b}) > 2(c/3)^{3/2}$ , there is no non-negative solution.*

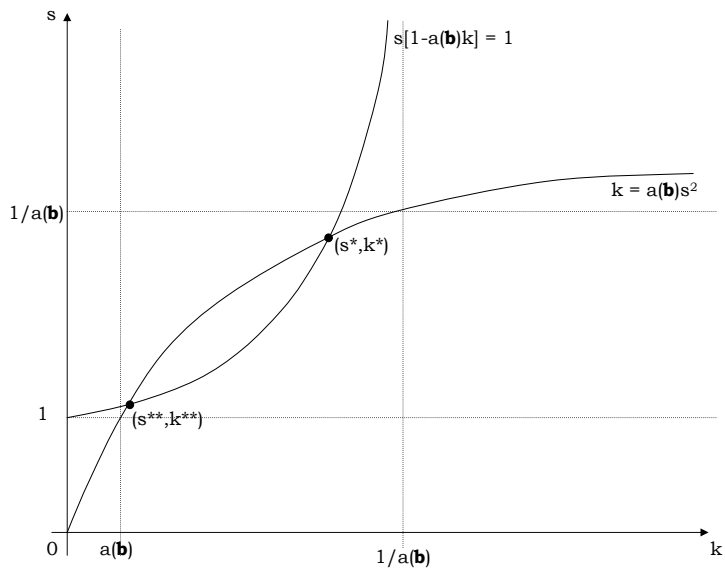
### 3.2 Pareto ranking of equilibria and the efficient outcome

Given an approximate equilibrium  $(\mathbf{s}^*, \mathbf{k}^*)$ , we denote by  $\mathbf{u}(\mathbf{s}^*, \mathbf{k}^*) = (u_1(\mathbf{s}^*, \mathbf{k}^*), \dots, u_m(\mathbf{s}^*, \mathbf{k}^*))$  the corresponding equilibrium payoffs. Denote also by  $(\mathbf{s}^E, \mathbf{k}^E)$  the (approximate) efficient outcome, i.e., the one (almost) maximizing the sum of payoffs for all players in a large  $m$ -replica game.

The next result compares equilibrium actions and payoffs across the two approximate equilibria characterized in Theorem 1.<sup>12</sup>

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<sup>12</sup>Notice that all the claims to follow are made on approximate equilibria rather than on the exact Nash equilibria of the game. Given the local continuity of the equilibrium correspondence around interior equilibria, the claims about



**Figure 1.**

Figure 1: Equilibrium conditions

**PROPOSITION 1** *Let  $(\mathbf{s}^*, \mathbf{k}^*)$  and  $(\mathbf{s}^{**}, \mathbf{k}^{**})$  be the two different approximate equilibria of an  $m$ -replica game . Then, without loss of generality,  $(\mathbf{s}^*, \mathbf{k}^*) \geq (\mathbf{s}^E, \mathbf{k}^E) \geq (\mathbf{s}^{**}, \mathbf{k}^{**})$  and  $\mathbf{u}(\mathbf{s}^E, \mathbf{k}^E) \geq \mathbf{u}(\mathbf{s}^*, \mathbf{k}^*) \geq \mathbf{u}(\mathbf{s}^{**}, \mathbf{k}^{**})$ , where  $\geq$  is the component-wise ordering.*

In words, the equilibrium actions are ranked component-wisely and the equilibrium payoffs are Pareto-ranked accordingly. From now on, we refer to the Pareto-superior and to the Pareto-inferior approximate equilibrium as the high and the low equilibrium, respectively. The socially efficient outcome lies in between the two equilibria, so that we can effectively talk about a too-high and a too-low equilibrium.

Denote by  $(s^*, k^*)$  and  $(s^{**}, k^{**})$  the baseline socialization and productive efforts that solve (6), and that enter the calculation of the high and the low equilibrium, respectively. Then, it is already apparent from Figure 1 that  $s^*/k^* \geq s^{**}/k^{**}$ . That is, the level of socialization per unit of productive investment is higher at the high equilibrium. At the high equilibrium, a high joint socialization effort creates tight links across players which then invest heavily in productive effort to build high cross synergies on this fertile ground. At the low equilibrium, low joint socialization efforts lead to a loose interaction pattern which hampers the scope for cross synergies, and thus the level of private investments. As a matter of fact, the synergistic multiplier is higher in the high equilibrium, and pulls up the level of production investments compared to the low equilibrium.

The equilibrium multiplicity identified in Theorem 1 reflects an inter-twinned coordination problem in the socialization process and in the production technology. These two coordination problems are rooted separately on the payoffs strategic complementarity both in socialization effort and in productive investment. The dependence of the cross returns in production on the population socialization profile,  $\partial^2 u_i(\mathbf{s}, \mathbf{k}) / \partial k_i \partial k_j = a g_{ij}(\mathbf{s})$  for  $i \neq j$ , relates these two coordination problems with each other, as reflected by the endogenous dependence of the synergistic multiplier (7) on the baseline socialization effort.

## 4 Socialization and investment

Recall that  $a(\mathbf{b})$  is a compound index of the technological synergy parameter and of the second-order average type  $\sum_{\tau=1}^t b_\tau^2 / \sum_{\tau=1}^t b_\tau$ , that measures the population heterogeneity in private returns to productive investment. This compound index enters directly in the equilibrium behavior (6) and, in particular, in the value of the synergistic multiplier (7).

Exogenous changes in the value of the technological parameter and/or in the group characteristics have an impact on socialization and investment that is channeled through the variations in  $a(\mathbf{b})$  that follow these changes. The next result clarifies how socialization and investment react to changes in  $a(\mathbf{b})$ .

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approximate equilibria are generally portable to properties of Nash equilibria, although some qualifications may be sometimes required.

**PROPOSITION 2** *Suppose that  $a(\mathbf{b})$  increases. Then, in both approximate equilibria of the replica game, the percentage change in socialization effort is higher than that of productive investment for all agents. Besides, the baseline equilibrium actions that solve (6) both increase at the low equilibrium and decrease at the high equilibrium.*

Recall that equilibrium actions are multiplicative in own traits,  $(s_i^*, k_i^*) = b_{\tau(i)}(s^*, k^*)$ , where  $(s^*, k^*)$  are the baseline socialization and productive investments that solve (6). Proposition 2 implies that the socialization effort per unit of productive investment,  $s_i^*/k_i^*$ , decreases at the high equilibrium and increases at the low equilibrium when  $a(\mathbf{b})$  goes up. It also states that the elasticity of socialization with respect to productive investment, that accounts for the relative variation of these two actions, is smaller than one at all equilibria.

At the high equilibrium, where socialization is overwhelming, agents substitute an increase in  $a(\mathbf{b})$  by an endogenous decrease in baseline socialization  $s^*$ . As a result, the synergistic multiplier decreases and the baseline productive investment is pulled down. These decrease in production feeds back and further dampens the level of socialization. And so forth, until this chain of cross influences sets at a new equilibrium. It turns out that socialization is more responsive than productive investment.

At the low equilibrium, instead, where the interaction pattern is diffuse, agents complement an increase in  $a(\mathbf{b})$  by an increase in socialization, which triggers an upwards jump in baseline productive investment. Again, socialization is here more responsive than productive investment.

The reason for this over-responsiveness of socialization can be better understood by inspecting the first order condition for socialization:

$$s = a(\mathbf{b})k^2$$

which implies that:

$$\frac{1}{s} \frac{\partial s}{\partial a(\mathbf{b})} = \frac{k^2}{s} + 2 \frac{1}{k} \frac{\partial k}{\partial a(\mathbf{b})} > \frac{1}{k} \frac{\partial k}{\partial a(\mathbf{b})}$$

In words, the fact that the marginal benefit of socialization effort is of the order of the square of  $k$ , whereas the marginal benefit is of the order of  $s$  is what drives this finding. This result, as we mentioned in the introduction, has interesting implications to understand the explosive increase in scientific collaborations as well as the large decreases of social capital in many developed societies.

Figure 2 illustrates how, at the low equilibrium, an increase in  $a(\mathbf{b})$  leads to an increase in both socialization and investment. At the high equilibrium, instead, baseline socialization and investment decrease.

Changes in the compound index  $a(\mathbf{b})$  can subsume various effects of very different nature. More precisely, this compound index responds positively to upwards shifts in the technological scale of synergistic returns  $a$ , to changes in the population that increase the second-order average type  $\sum_{\tau=1}^t b_{\tau}^2 / \sum_{\tau=1}^t b_{\tau}$ , and to combination of both effects.

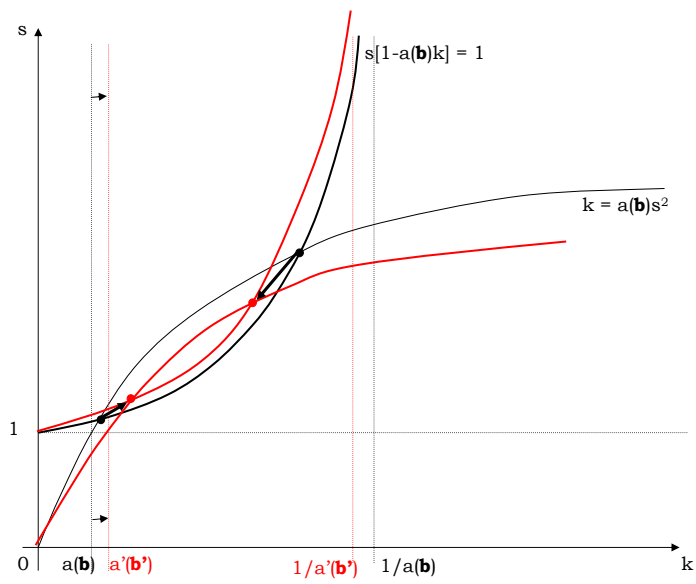


Figure 2.

Figure 2: Equilibrium comparative statics

For instance, a mean-preserving spread in idiosyncratic traits leaves  $\sum_{\tau=1}^t b_\tau$  constant while  $\sum_{\tau=1}^t b_\tau^2$  grows. The ensuing upwards shift in the second-order average type increases  $a(\mathbf{b})$ .

Suppose also that  $a$  and all the  $b_\tau$ 's are all scaled up by a common factor. In particular, consider the following variation of payoffs (4), where  $\lambda > 0$ :

$$u_i(\mathbf{s}, \mathbf{k}) = b_{\tau(i)} k_i + a \sum_{j=1, j \neq i}^n g_{ij}(\mathbf{s}) k_j k_i - \frac{\lambda}{2} c k_i^2 - \frac{\lambda}{2} s_i^2, \quad (8)$$

The game with payoffs (8) has the same equilibria than the game with original payoffs (4), where the types  $b_\tau$  and the synergy scale parameter  $a$ , and thus the compound index  $a(\mathbf{b})$ , are all scaled homothetically by  $1/\lambda$ .

Proposition 2 encompasses all those cases and many others, and pins down the relative variation of socialization and productive investment, and the absolute variation of baseline socialization and productive investment, for all these multifarious changes in parameters.

Notice, however, that equilibrium actions  $(s_i^*, k_i^*) = b_{\tau(i)}(s^*, k^*)$  need not move in the same direction than baseline equilibrium actions  $(s^*, k^*)$ . This may be so at the low equilibrium when the changes in  $a(\mathbf{b})$  result from variations in parameter values for which  $b_{\tau(i)}$  moves in the opposite direction than  $a(\mathbf{b})$ , and at the high equilibrium if  $b_{\tau(i)}$  moves in the same direction than  $a(\mathbf{b})$ .

The following result clarifies this point.

**COROLLARY 1** *An increase in  $a(\mathbf{b})$  for which  $b_\tau$  does not decrease (resp. does not increase) increases (resp. decreases) the socialization and investment efforts of all type  $\tau$  players at the low equilibrium (resp. at the high equilibrium).*

## 5 Population composition and welfare

We now document the comparative statics of individual and aggregate equilibrium payoffs when  $a(\mathbf{b})$  varies, with a particular emphasis on the effect of the group heterogeneity.

### 5.1 Welfare

When  $m$  gets large, approximate equilibrium payoffs corresponding to baseline efforts  $(s^*, k^*)$  that solve (6) are given by the following expression:

$$u_i^* = \frac{b_{\tau(i)}^2}{2a(\mathbf{b})} \frac{s^*}{k^*} + o(1) = \frac{b_{\tau(i)}^2}{2} k^* + o(1), \text{ for all } i = 1, \dots, mt. \quad (9)$$

Given the expression for equilibrium payoffs (9), the comparative statics of equilibrium actions established in Proposition 2 and Corollary 1 above have straight implications for the reaction of individual well-being to changes in the compound index  $a(\mathbf{b})$ .

**PROPOSITION 3** *An increase in  $a(\mathbf{b})$  for which  $b_\tau$  does not decrease (resp. does not increase) increases the low equilibrium payoffs (resp. decreases the high equilibrium payoffs) of all type  $\tau$  players.*

In particular, an increase in the synergy parameter  $a$  induces a downwards shift of equilibrium payoffs at the high equilibrium, and an upwards shift at the low equilibrium. Indeed, scaling up the scope for synergies helps alleviating the coordination features of both equilibria. At the low equilibrium, characterized by an under-provision of production and socialization efforts, the interaction pattern is tightened and investments increment. At the high equilibrium, instead, where over-provision prevails, the social network is loosened and investments reduced.

Proposition 3 also documents the changes in individual welfare following a change in the population distribution of types. We explore some changes in the types profile  $\mathbf{b} = (b_1, \dots, b_t)$  that induce an increase in  $a(\mathbf{b})$ .

Consider two type profiles  $\mathbf{b}' \geq \mathbf{b}$ , where  $\geq$  is the component-wise ordering. The marginal private returns to productive investment of every agent are not smaller under  $\mathbf{b}'$  than under  $\mathbf{b}$ , and are strictly higher for at least one type when  $\mathbf{b}' \neq \mathbf{b}$ . Even though  $\mathbf{b}'$  is obtained from  $\mathbf{b}$  by increasing the type of some agents, it is not always true that the value of  $a(\mathbf{b}')$  so obtained is higher than that of  $a(\mathbf{b})$ . The following results clarify this point.

Recall that  $0 < b_1 \leq b_2 \leq \dots \leq b_t$ .

**PROPOSITION 4** *Let  $\mathbf{b} = (b_1, \dots, b_t)$  be a type profile such that  $2b_1 \geq b_t$ . Then, the low equilibrium payoffs of every player increase with  $b_\tau$ , while the high equilibrium payoffs of every player of type  $\tau' \neq \tau$  decrease with  $b_\tau$ , for all  $\tau \in T$ .<sup>13</sup>*

Recall that, at equilibrium, all agents have the same distribution of types in their social circle, and interactions with a type  $\tau$  represent a fraction  $b_\tau / \sum_{\tau' \in T} b_{\tau'}$  of the agent's total socialization intensity. When  $2b_1 \geq b_t$ , the ratio of higher to lower type values is never higher than two. As a result, the type composition of agent's social circles are moderately uneven, and high types are not over-represented with respect to lower types.

With a moderately uneven type distribution, agents of different types contribute in not too disparate proportions to the creation of cross-synergies.

More precisely, the synergistic payoff to a bilateral interaction between players  $i$  and  $j$  is  $ag_{ij}(\mathbf{s})k_i k_j$ . At equilibrium, we can rewrite this spillover as  $as_i^* s_j^* k_i^* k_j^* / \sum_{\tau \in T} b_\tau s^*$ . Holding the average type constant, the two partners contribute, respectively,  $s_i^* k_i^*$  and  $s_j^* k_j^*$ , to the creation of the synergy payoffs. Relative contributions to the synergy are thus equal to  $b_{\tau(i)}^2 / b_{\tau(j)}^2$ . Two interacting partners with identical type contribute symmetrically to the cross spillover they exert on each other, while two interacting partner with different type contribute asymmetrically. When

<sup>13</sup>A weakening of the sufficient condition  $2b_1 \geq b_t$  is to assume that  $b_\tau^2 + 2b_\tau \sum_{\tau' \neq \tau} b_{\tau'} \geq 2 \sum_{\tau' \neq \tau} b_{\tau'}^2$ , for all  $\tau \in T$ .

$2b_1 \geq b_t$ , these asymmetric contributions are not too strikingly different, with values in  $[1/4, 4]$ , and cross spillovers are moderately asymmetric.

With moderately asymmetric cross spillovers, any type value unilateral increase benefits all players at the low equilibrium who each increase the spillovers they exert to every other interacting partner.

Instead, when type values differ drastically with each other, players' contributions to the spillovers with players of a different type are highly asymmetric. An increase in the low type value can now decrease the welfare of high type players at the low equilibrium, as illustrated below.

**EXAMPLE 1** *Consider two types  $\mathbf{b} = (b_1, b_2)$  with  $b_1 < (\sqrt{2} - 1)b_2$ . Then,  $a(\mathbf{b})$  is a strictly decreasing function of the low type  $b_1$ .<sup>14</sup> An increase in  $b_1$  decreases the payoffs of the high type players at the low equilibrium.*

When  $b_1 < (\sqrt{2} - 1)b_2$ , the fraction of type 2 players in every individual social circle is bounded from below by  $1/\sqrt{2}$ , a lower bound of roughly 70, 7% interactions with type 2 players. In particular, high type 2 players interact overwhelmingly with players of the same type. This in-breeding bias is accompanied by symmetric cross-spillovers across high type players to which they all contribute intensively. High type players also contribute asymmetrically to spillovers they exert on low type players. But the interactions sustaining these spillovers are rare, and they main concern is the creation of within-type symmetric spillovers.

Suppose now that the value of  $b_1$  increases. The asymmetric high-low type interactions become more prevalent. A higher share of the socialization and production efforts of high type players is now beneficial to low type players, who partially free-ride on these spillovers to which they themselves don't contribute much. High type players then react to this increase in  $b_1$  by narrowing down the size of their social circles at the low equilibrium and, more generally, by decreasing the generation of cross-spillovers that they now share symmetrically among themselves in a smaller proportion.

The previous results characterize the effect on individual payoffs of changes in type composition. Now following a mean preserving effect in types, the compound index  $a(\mathbf{b})$  increases. This induces an upward shift of baseline actions at the low equilibrium. Individuals experiencing an increase in type will certainly experience an increase in payoffs, but the effect is ambiguous for agents experiencing a decrease in types. The next result shows, however, that the overall aggregate effect is positive. The argument is symmetric at the high equilibrium.

**PROPOSITION 5** *Aggregate payoffs increase at the low equilibrium and decrease at the high equilibrium following a mean-preserving spread of the population type profile  $\mathbf{b} = (b_1, \dots, b_t)$ .<sup>15</sup>*

<sup>14</sup>We have  $a(\mathbf{b}) = (b_1^2 + b_2^2) / (b_1 + b_2)$ . The sign of  $\partial a(\mathbf{b}) / \partial b_1$  is that of  $b_1^2 + 2b_1b_2 - b_2^2$ , which is negative when  $(b_1 + b_2)^2 < 2b_2^2$ , that is,  $b_1 < (\sqrt{2} - 1)b_2$ .

<sup>15</sup>That leaves  $\sum_{\tau=1}^t b_\tau$  constant but increases  $\sum_{\tau=1}^t b_\tau^2$ .

## 5.2 Community segregation

In section 5.1 we document the welfare consequences of changes in the type distribution of communities, for both individual types and for communities as a whole. Implicitly this mapping allows us to characterize the individual preference induced by equilibrium behavior and payoffs over the type composition to which the individual belongs. Our equilibrium analysis generates endogenous preferences over groups, embodied in the indirect utility function created by the equilibrium mapping.

Individual tastes over group composition irremediably pose the question of group formation.

Previous research on group formation typically consider individuals exogenously endowed with preferences over groups. For instance, in the seminal work of Becker (1957) and Schelling (1971), individuals with different idiosyncratic traits intrinsically like or dislike each other. Here, in contrast, the group externality results from equilibrium behavior. Other's idiosyncratic traits do not enter directly into the specification of an individual's utility function. They only do so at equilibrium, where individual actions subsume the group heterogeneity in a consistent manner.

Following this early literature, we analyse how preferences over groups affect group formation. Because we deal with endogenous preferences, our analysis sheds light on the impact of technological parameters (such as returns to private investment and the scope for building spillovers) on individual incentives to group formation and social welfare. This, in turn, opens the scope for the design and implementation of viable public policies based on observable technological parameters rather than on some exogenous (dis)taste for diversity. We also derive implications for interaction patterns (intra and between group socialization) for endogenously formed groups.

In our model, the preference over group composition depend on the compound index  $a(\mathbf{b})$ . More precisely, consider a group with a population type profile  $\mathbf{b} = (b_1, \dots, b_t)$ . Denote by  $\mathbf{b}^{(m)}$  the vector of length  $n = mt$  obtaining by replicating  $m$  times the population with type profile  $\mathbf{b}$ . Clearly  $a(\mathbf{b}) = a(\mathbf{b}^{(m)})$ . Consider an individual  $i$  with type  $b_i$  that joins this group. Now the population size is  $m + 1$  and the new type profile is  $(b_i, \mathbf{b}^{(m)})$ . When  $m$  is large enough it is clear that the compound index for this community of size  $m + 1$  can be written as:

$$a(b_i, \mathbf{b}^{(m)}) = a(\mathbf{b}) + o(1). \quad (10)$$

Therefore the payoff accruing to the new community member is:

$$u_i^* = \frac{1}{2} b_i^2 k^* (a(\mathbf{b})) + o(1), \quad (11)$$

where  $k^*(a(\mathbf{b}))$  is the baseline productivity that solves (6) when the vector of type profiles is exactly  $\mathbf{b}$ .

Communities with a higher  $a(\mathbf{b})$  thus lead to higher (resp. lower) payoffs for the newcomer at the low (resp. high) equilibrium compared to the source community. Also, the returns from changing community are proportional to the square of the type of the individual who moves,  $b_i^2$ .

An individual with a higher type gains more to a community with a higher  $a(\mathbf{b})$  than an individual with a lower type (at a low equilibrium, the situation is reversed at a high equilibrium). This differential sensitivity of low and high type movers to the composition of the community profile the individual belongs to has segregation-inducing effects that we now analyze.

Take two communities,  $\mathbf{b}$  and  $\mathbf{b}'$ . Consider an individual  $i$  who is pondering whether to move from  $\mathbf{b}'$  to  $\mathbf{b}$ . Using (11) the relative gains from moving are:

$$u_i^*(b_{\tau(i)}, \mathbf{b}) - u_i^*(b_{\tau(i)}, \mathbf{b}') = \frac{1}{2} b_{\tau(i)}^2 [k^*(a(\mathbf{b})) - k^*(a(\mathbf{b}'))] + o(1)$$

If a player of arbitrary type  $b_{\tau(i)}$  gains from the move, any player with any other type would also gain. However, the gains are higher for higher type players. This type-monotonicity in relative gains is what leads to an equilibrium with community stratification by types.

More precisely, consider a finite set of communities  $\ell \in \{1, \dots, L\}$ , each with  $n_\ell$  slots where  $\sum_{\ell=1}^L n_\ell \geq n = mt$  for a high enough  $m$ . We also assume that each  $n_\ell$  is high enough so that the compositional impact of changing one players' type on the compound index for community  $\ell$  is small, as in (10).

Order arbitrarily the available communities.

We denote by *top-down sorting* the following assignment of players into communities according to their type. Community 1 gets assigned the  $n_1$ -highest type players, community 2 the  $n_2$ -highest type players among the remaining ones, and so on until all players are assigned to one (and only one) community.

Recall that there are  $m$  players of each type. If  $m \geq n_1$ , the top-down sorting assignment produces a homogeneous community 1 that contains only individuals with the highest type. Instead, if  $2m \geq n_1 > m$ , community 1 contains all the individuals with the highest type available in the population, and some (exactly  $m - n_1$ ) individuals with the next highest type.<sup>16</sup> And so on.

The top-down sorting leads to a segregated community structure with types stratified from higher to lower. Namely, given two communities  $\ell > \ell'$  and two players  $i, i'$  that are assigned to either community by top-down sorting, then  $b_{\tau(i)} \geq b_{\tau(i')}$ . To ensure that this inequality is strict for at least one pair of players in two different communities, we assume that two successive communities cannot be fully occupied by players of the same type, that is,  $m > n_\ell + n_{\ell+1}$  for all  $\ell = 1, \dots, L - 1$ .

We define similarly an assignment that allocates successively individuals starting from low types and moving on to higher ones that we refer to as *bottom-up sorting*.

To join a neighborhood  $\ell$ , individuals must pay a fee  $p_\ell$  to the owner of the community  $\ell$ . We say that an assignment of individuals to communities and a vector of community prices forms an equilibrium when, given the prices, no individuals prefers to change communities and either a community is full or its associated fee is zero.

<sup>16</sup>More generally, let  $k$  be an integer such that  $(k + 1)m \geq n_1 > km$ . Then, Community 1 contains all the  $m$  type  $t, \dots, t - k + 1$  players, and  $n_1 - km$  type  $t - k$  players chosen arbitrarily among the  $m$  type  $t - k$  players available in the population.

**PROPOSITION 6** *There exists an assignment equilibrium with top-down sorting where individuals play their low equilibrium actions for their community. Denote by  $\mathbf{b}^\ell$  the type profile in community  $\ell$  under top-down sorting, and by  $\underline{b}^\ell$  the lowest type in this community. Then, the fee for a full community  $\ell$  is:*

$$p_\ell = \frac{1}{2} \underline{b}^{\ell 2} \left[ k^* \left( a \left( \mathbf{b}^\ell \right) \right) - k^* \left( a \left( \mathbf{b}^{\ell+1} \right) \right) \right], \ell = 1, \dots, L - 1, \quad (12)$$

and  $p_L = \frac{1}{2} \underline{b}^{L 2} k^* \left( a \left( \mathbf{b}^L \right) \right)$ . *Symmetrically, there exists an assignment equilibrium with bottom-up sorting where individuals play their high equilibrium actions for their community; the fee for a full community  $\ell$  under bottom-up sorting is computed with (12) where  $\underline{b}^\ell$  is replaced by the highest type in community  $\ell$ .*

It would not be very hard to reformulate the model so that segregation is not complete, but only partial, in that the cross communities interactions are still allowed although at lower intensities than intra-community ones.

**PROPOSITION 7** *Social welfare is higher with the top-down segregated low equilibrium than with a fully integrated low equilibrium community.*

Recall that segregation at the low equilibrium is sponsored by the high types. Instead, segregation at the high equilibrium is sponsored by the low types. These equilibria reflect fundamental coordination problems arising from externalities. In one of them, there is underinvestment in both productive effort and social capital. In the other one, there is overinvestment. In the low equilibrium there is underinvestment. Notice that all individuals benefit symmetrically from the synergy in every bilateral interaction in which they are involved. However, despite the symmetric benefits, contributions to the synergies are very different for different types. In this way, the high types are relatively more harmed from the underinvestment. They would benefit more from a change in the environment that increased the number of high contributors, than the low contributors would be harmed from a decrease in the share of high contributors in their neighborhood. Thus a new equilibrium with segregation would increase total social surplus. Proposition 5 already hints at this kind of convexity, that is, the fact that social welfare decreases with mixity at the low equilibrium.

The literature on the productivity effects of “economic density”<sup>17</sup> has concentrated on agglomeration externalities to explain why denser economic areas are more productive. The models in that literature typically assume away individual heterogeneity and its impact in the observed productivity differentials by location. Combes, Duranton and Gobillon (2007), however, note that “individual skills account for a large fraction of existing spatial wage disparities with strong evidence of spatial sorting by skills.”<sup>18</sup> An advantage of our model is that it incorporates a particular

<sup>17</sup>See Ciccone and Hall (1996) for a seminal contribution. Duranton and Puga (2004) survey this literature.

<sup>18</sup>Andersson, Burgess and Lane (2007) make a similar point.

form of agglomeration externalities as well as individual heterogeneity. This combination, in turn, allows us to explain the pattern of sorting into different locations by differently able individuals, and therefore the wage disparities that go along with productivity differentials.

By placing the individual heterogeneity in a parameter clearly identifiable with productivity we can model the effect of policies that impact individual productive effort into segregation. These turn out to have different properties, in general, from those of other models that have been explored in the literature for similar problems.

For example, the skill complementarities in our model are endogenous. This differentiates it from other models of type complementarity, like the one in Kremer and Maskin (1996). Our agents care about the type of others only because of the effect that type has on effort. Thus, a policy which subsidized the cost of production of agents with a lower  $b_i$  would have an immediate desegregating effect (whereas it would have no impact in a world à la Kremer and Maskin 1996).

Also, contrary to the model of Bénabou (1996) which has no type heterogeneity, de-coupling local investment in education from local income taxes need not change the amount of segregation in our model. To see this, suppose that individuals are initially different in their  $b_i$ 's, but that investment in human capital can raise it (or lower costs). Even if all individuals invest the same in their  $b_i$ 's, segregation would still occur. In fact, in a model with this effect of human capital on  $b_i$ , any investment that is neutral or favors the high types, would *increase* segregation, not decrease it (with respect to a situation with no investment at all)..

## 6 The topology of social interactions

In our model local externalities come from two sources. On the one hand, the pure productive synergistic return  $k_i k_j$ . On the other hand, the intensity of the local interaction  $g_{ij}$  which allows this synergy to take effect. Up to now, we have thought of the social interaction variable  $s_i$  as the intensity of a bilateral relationship. One could also interpret social interaction intensities as the probability of crystallizing a particular bilateral relationship.<sup>19</sup>In other words, we could view a realized meeting as a link between the two involved parties, activated with some probability  $g_{ij}$ . In this way, the whole pattern of social interactions can be viewed as a random graph, where each link  $ij$  is activated with independent probability  $g_{ij}$ , known as the multinomial random graph model.

Viewing social interactions through the lens of random graph theory brings a number of interesting implications. In particular, one can discuss a number of topological regularities of the social interaction pattern, such as the distribution of individual connectivities, the network internal community structure, and confront them with known empirical regularities for these variables. More importantly, in our case, the parameters that determine the random graph model are an equilibrium outcome that reflect individual incentives. Therefore our model sheds light on the

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<sup>19</sup>This interpretation, of course, require that in equilibrium  $g_{ij} \leq 1$ .

interplay between network formation and the private economic use of this jointly created device, and connects topological features of the network to individual behavior and payoffs.

For all  $\mathbf{x} \in \mathbb{R}_+^t$ , define  $\bar{\mathbf{x}} = \sum_{\tau=1}^t x_\tau/t$ , and  $v(\mathbf{x}) = \sum_{\tau=1}^t x_\tau^2/t - \bar{\mathbf{x}}$ . These are, respectively, the average and the empirical variance of the coordinates of  $\mathbf{x}$ . We extend this definition to any non-negative vector in an Euclidean space of finite arbitrary size.<sup>20</sup>

Consider an approximate equilibrium  $(\mathbf{s}^*, \mathbf{k}^*)$  of the  $m$ -replica game, that corresponds to some solution  $(k^*, s^*)$  to (6).

Theorem 1 and (6) imply that the distribution of socialization efforts  $g_i(\mathbf{s}^*) = s_i^* = s^* b_{\tau(i)}$  is related, at equilibrium, to the distribution of types  $\mathbf{b} = (b_1, \dots, b_t)$ . In equilibrium each link  $ij$  is formed with independent probability  $g_{ij}(\mathbf{s}^*)$ . This random graph has therefore an expected connectivity sequence  $(s_1^*, \dots, s_n^*)$ . We can map the population ex ante heterogeneity into the connectivity distribution for the equilibrium random graph.

More precisely, the average connectivity is  $\bar{\mathbf{s}}^* = s^* \bar{\mathbf{b}}$ , proportional to the average type (a compositional effect) and to the baseline socialization level (a level effect). However, at equilibrium, the baseline socialization is affected by the type profile. Thus, the individual heterogeneity impacts the average socialization level both directly and indirectly. In particular, if we consider a top-down segregated low equilibrium, the overall average connectivity differs from that of a fully integrated society.

Also, the empirical variance of connectivities is  $v(\mathbf{s}^*) = s^{2*} v(\mathbf{b})$ . Therefore,

$$\frac{\sqrt{v(\mathbf{s}^*)}}{\bar{\mathbf{s}}^*} = \frac{\sqrt{v(\mathbf{b})}}{\bar{\mathbf{b}}}. \quad (13)$$

The heterogeneity in connectivities in the resulting equilibrium network is thus solely driven by the heterogeneity in private returns, and is thus the same for both equilibria. Increasing the latter increases the former, and by varying  $\mathbf{b}$  adequately we can cover a broad range of random graph topologies, including the heavy tailed connectivity distributions identified in the data.<sup>21</sup> For instance, a mean-preserving spread in  $\mathbf{b}$  increases  $\sqrt{v(\mathbf{s}^*)}/\bar{\mathbf{s}}^*$ , and leads to a connectivity distribution with a fatter tail.

Beyond this comparative statics about ratios (13), Proposition 2 allows to conduct comparative statics directly about the average connectivity  $\bar{\mathbf{s}}^*$  and the variance  $v(\mathbf{s}^*)$ .

More precisely, a mean-preserving spread in private returns  $\mathbf{b}$  has an indirect effect on the average connectivity (through the change in  $s^*$ , the solution to (6)), and both a direct and an indirect effect on the variance of connectivities (through both the change in  $s^*$  and the change in  $v(\mathbf{b})$ ). At the low-action equilibrium, both the variance and the average connectivity increase. At

<sup>20</sup>For all  $\mathbf{x} \in \mathbb{R}_+^n$ , we set  $\bar{\mathbf{x}} = \sum_{i=1}^n x_i/n$ , and  $v(\mathbf{x}) = \sum_{i=1}^n x_i^2/n - \bar{\mathbf{x}}$ , for all  $n \geq 1$ .

<sup>21</sup>The distribution of network connectivities tends to be fat tailed for some networks (see Jackson and Rogers 2007 for amendments to this claim). That is, there is in some networks a much higher proportion of network nodes with a high number of network links than if the network links were created uniformly and independently at random. Notice, however, that a fat tailed degree distribution requires a fat tailed distribution of population traits.

the high-action equilibrium, instead, the average connectivity decreases while the impact on the variance remains ambiguous. Invoking Proposition 5, we can conclude that an increase in both the variance and the average connectivity is concomitant to a decrease in total welfare. Instead, a decrease in average connectivity together with an increase in the variance of connectivities comes together with an increase in total welfare (provided, of course, that the variations in the network topology result from a mean-preserving spread in private returns).

Note that an increase in  $a$  alone also affects the variance and the average connectivity through the resulting impact on  $s$ .

More generally, an increase in  $a(\mathbf{b})$  increases the variance and the average connectivity at the low-equilibrium, and decreases the average connectivity at the high-equilibrium, although it may have an ambiguous impact on  $\sqrt{v(\mathbf{s}^*)/\mathbf{s}^*}$ .

Our static model also allows to draw conclusions on the average distance in the equilibrium random network.<sup>22</sup> Following Chung and Lu (2002) the average distance in a random graph with given expected connectivity  $(s_1^*, \dots, s_n^*) = s^*(b_{\tau(1)}, \dots, b_{\tau(n)})$  is given by:<sup>23</sup>

$$(1 + o(1)) \frac{\log(mt)}{\log(s^*\mathbf{b})}.$$

This average distance increases slowly with the population size  $mt$ .

When the level of synergistic returns  $a$  increases while private returns  $\mathbf{b} = (b_1, \dots, b_n)$  remain constant, the average distance decreases at the low equilibrium while equilibrium payoffs increase. Instead, the average distance increases at the high equilibrium together with a decrease of equilibrium payoffs.

Suppose now that  $a$  and  $\mathbf{b}$  change homothetically, and are all scaled up by a common parameter. We still have a decrease of the average distance coupled with increasing payoffs at the low equilibrium. The impact of this exogenous change of parameters at the high equilibrium is now ambiguous. However, an increase of the average distance can only happen at this high equilibrium, and is then concomitant with a decrease in payoffs for all agents.

The impact of a mean-preserving spread in private returns on the average distance is similar to a change in the level  $a$  of synergies. That is, the average distance goes down and the average payoffs go up at the low equilibrium, and reciprocally at the high equilibrium (average distance up and average payoffs down).

The next table summarizes this discussion.

Table 2. Comparative statics in the low and high equilibrium

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<sup>22</sup>The average distance (or shortest path) between two network nodes is very small compared to the network size, and grows very slowly with this size in many empirically observed networks. For instance, the network of actors who have acted together in at least one Hollywood movie comprises 225,226 individuals and has an average path length of 3.65. See Albert and Barabasi (2002).

<sup>23</sup>Provided that link intensities are all smaller than one, and under some additional technical conditions.

	Low equilibrium					High equilibrium				
	$\bar{s}$	$v(\mathbf{s})$	Clust.	$\overline{\text{dist.}}$	payoffs	$\bar{s}$	$v(\mathbf{s})$	Clust.	$\overline{\text{dist.}}$	payoffs
$a$ up	+	+	+	−	+	−	−	−	+	−
$(a, \mathbf{b})$ all up	+	+	+	−	+	·	·	·	·	−
$\mathbf{b}$ spread	+	+	+	−	+	−	·	·	+	−
					(total payoffs)					(total payoffs)

Our static model of network formation also delivers implications for the convolutedness of social interaction. An observational measure for this is probability with which two given linked nodes are linked to a common third-party. This is called the clustering coefficient. Take three nodes  $i, j, l$  such that  $i$  and  $j$  are linked, and so are  $j$  and  $l$ . Then, in our model, the probability that  $i$  and  $l$  are linked when  $m$  is high is roughly  $g_{il}(\mathbf{s}^*)$ , independent of the links  $ij$  and  $jl$  and of the order of the inverse of population size. More precisely, an approximate expression for the size of the clustering is:<sup>24</sup>

$$\frac{1}{mt} \frac{s^*}{\bar{\mathbf{b}}} \left( 1 + \frac{v(\mathbf{b})}{\bar{\mathbf{b}}} \right)^2. \quad (14)$$

Notice that this number is inversely proportional to population size  $mt$ . In some large real-life networks, though, the actual clustering coefficient is much higher. For example the clustering coefficient of the movie actor network is almost 3,000 times higher than that of a purely random network with identical average connectivity. However, equation (14) suggests that small circles of interaction deliver levels of clustering similar to those observed in the data. Segregation into communities of moderate size thus helps accommodating the model predictions with moderate to high clustering levels.

Finally, empirically observed social networks sometimes display a giant component, that is, they contain a subnetwork including a huge share of the population and for which there is a path inside it connecting any two players in this population share.<sup>25</sup> In the section 7.2 below, we provide conditions on the exogenous parameters of the model for the emergence of a giant component for the case of an homogeneous populations,  $b_1 = \dots = b_t$ , and general costs.

## 7 The case of homogeneous populations

Proposition 6 implies that individuals of the either extreme type prefer a society composed only of individuals such as themselves to any other mixture. Provided that the institutional environment

<sup>24</sup>This clustering coefficient is computed averaging over all players, the percentage of triangles they are involved in, out of the total potential triangles.

<sup>25</sup>However, giant components are not always the rule. For instance, the biggest component of a network of scientific collaborations among academic economists analyzed by Goyal, Moraga-González and van der Leij (2005) comprises 33,027 authors out of a total of 81,217 authors, that is, 40.7 percent of the total population.

allows them to segregate themselves, they will form a separate society. One should thus expect that at least some homogeneous groups would exist in a given society. However, given that at least some of the types within a subgroup would lose from the segregation of the opposite types, it should not be excluded that the society would not leave complete freedom for segregation at all levels, and heterogeneous group may anyway form.<sup>26</sup>

In what follows, we concentrate our attention on homogeneous groups, for which we can conduct some robustness checks on the technology and for which further insights on the topology are possible.

## 7.1 Equilibrium analysis and comparative statics

We now consider an homogeneous population of players with a single type  $b$ , but allow for non-linear marginal costs of both socialization and investment. Player  $i$ 's utility is:

$$u_i(\mathbf{s}, \mathbf{k}) = bk_i + a \sum_{j=1, j \neq i}^n g_{ij}(\mathbf{s}) k_i k_j - \frac{1}{d+1} k_i^{d+1} - \frac{1}{d+1} s_i^{d+1}, \quad (15)$$

where  $a, b \geq 0$  and  $d \geq 1$ . The case  $d = 1$  corresponds to quadratic costs. As  $d$  increases, the cost function becomes steeper both for socialization efforts and for productive investments.<sup>27</sup>

We focus on symmetric equilibria, where all (homogeneous) players choose the same socialization effort and productive investment. When the resulting link intensity is smaller than one, such symmetric equilibria give rise to random graphs where the probability of link creation is independent and identical across all links. This Bernoulli process of link formation was first analyzed by Erdős and Rényi (1959), who provides a number of interesting topological properties for large Bernoulli networks, that is, as the number of the nodes tends to infinity. In particular, when the population gets large, the Erdős and Rényi random graph induces a Poisson distribution over node connectivities.

We analyze the topology and welfare properties of Poisson-Nash networks for the class of cost functions defined in (15).

We first introduce some notations.

For all  $\alpha < \beta$ , define:

$$\mathbf{1}_{(\alpha, \beta]}(x) = \begin{cases} 1, & \text{if } \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}.$$

We introduce also the following function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  given by:

$$\phi(x) = d^{\frac{1}{d+1}} \left[ x^{d+1} - b^{1+\frac{1}{d}} \right]^{\frac{1}{d+1}}.$$

We are now ready to state the main characterization result.

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<sup>26</sup>Of course, it is also possible that there are technological restrictions, such as increasing returns at certain levels, which would make very small groups inefficient.

<sup>27</sup>Compared to (4), we also consider interchangeable cost structures for socialization and productive investment, i.e.,  $c = 1$ , a simplification that does not alter our qualitative conclusions.

**THEOREM 2** *Suppose that  $\frac{d^2}{2+d^2} \left(\frac{2}{2+d^2}\right)^{\frac{2}{d^2}} > a^{1+\frac{1}{d}} b^{\frac{2}{d^2}}$ . Then, as the population size increases, the pure strategy Nash equilibrium strategies converge to the solutions to:*

$$\begin{cases} s^d = \mathbf{1}_{(-\infty, \phi(k)]}(s) ak^2 \\ k^d [1 - as] = b \end{cases} . \quad (16)$$

*There is always one partially corner equilibrium, and up to two different interior equilibria.*

Notice that  $(s^*, k^*) = (0, b^{1/d})$  is always a solution to (16), which corresponds to the partially corner equilibrium identified in Lemma 2 for the case of quadratic costs,  $d = 1$ , a symmetric cost structure,  $c = 1$ , and an homogeneous population. The corresponding equilibrium payoffs are  $db^{1+1/d}/(d+1)$ . At this equilibrium, the network of synergies is empty.

For large enough populations, we can also have up to two *interior symmetric equilibria*. Inspecting (16), these interior equilibria  $(s^*, k^*)$  solve:

$$\begin{cases} s^d = ak^2 \\ k^d (1 - as) = b \end{cases} , \quad (17)$$

which generalizes (6) for arbitrary  $d$  with the added condition that the equilibrium strategies  $(s^*, k^*)$  are such that  $s^* \leq \phi(k^*)$ . In fact, it can be readily checked that this last inequality is equivalent to:

$$\frac{1}{d+1} \left( dk^{*d+1} - s^{*d+1} \right) = u_i(s^*, k^*) > u_i(b^{1/d}, 0) = \frac{d}{d+1} b^{1+\frac{1}{d}}. \quad (18)$$

In words, the condition  $s^* < \phi(k^*)$  guarantees that  $(b^{1/d}, 0)$  is not a strict best-response by some arbitrary player  $i$  to the others playing  $(s^*, k^*)$ .<sup>28</sup>

The fact that the inequality (18) holds for any interior Nash equilibrium implies that any interior equilibrium Pareto dominates the partially corner equilibrium where players do not socialize and invest the optimum level  $b^{1/d}$  that corresponds to their private costs and returns.

Suppose now that the exogenous parameter values are such that two interior equilibria exist. It turns out that their corresponding actions can also be ranked unambiguously. As in Proposition 1, we can thus speak of a low-actions (interior) equilibrium and a high-actions (interior) equilibrium for large enough populations. The response to these equilibrium actions to incentives is identical to that documented in Proposition 2.

**PROPOSITION 8** *Suppose that (17) has three different solutions. Let  $(s^*, k^*) \geq (s^{**}, k^{**})$  be the two ranked interior symmetric approximate equilibria for a large enough population. When  $a$  increases, the low equilibrium actions  $(s^{**}, k^{**})$  increase, while the high equilibrium actions  $(s^*, k^*)$  decrease. In both cases, the percentage change in socialization effort is higher than that of productive investment (in absolute values).*

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<sup>28</sup>The conditions on the exogenous parameter values imposed in Theorem 2 guarantee that the first-order conditions for interior equilibria (17) have at least one solution, and that the second-order conditions hold at these points.

Notice that the previous result implies that the ratio of socialization to production  $s/k$  is an increasing function of  $a$  at equilibrium. Factorizing by  $s^{*d+1}$  in the expression for equilibrium payoffs given by (18), one can readily conclude that individual payoffs decrease with  $a$  at the high equilibrium.

## 7.2 The topology of Erdős-Rényi equilibrium networks

In the Erdős-Rényi (Bernoulli) random graphs that correspond to the interior and symmetric Nash equilibria of Theorem 2, the expected number of links to each player is  $s^*$ , and each potential link in the network is created with (approximately) independent probability  $s^*/n$  when the population gets large. The fact that link creation is *i.i.d.* implies, in particular, that the network connectivity (or degree) is not correlated across different nodes.

Beyond this vanishing degree-degree correlation across nodes as the population gets larger, large Erdős-Rényi networks display a number of interesting topological features.

For instance, when  $s^* < 1$ , realizations of the random graph are composed of a huge number of disjoint small trees with very high probability. We say that the network is *fragmented*. Instead, when  $s^* > 1$ , realizations of the random graph display a single *giant component* that encompasses a high fraction of all the network nodes with very high probability.<sup>29</sup>

The next result ties down the existence of a giant component to conditions on the exogenous parameter values of the model.

**PROPOSITION 9** *When  $a \geq 1$ , no equilibrium network has a giant component. Suppose that  $a < 1$  and that there are two non-empty equilibrium networks. Then, the high equilibrium network has a giant component while the low equilibrium is fragmented if and only if  $ab^{2/d} < (1 - a)^{2/d}$ . If, instead,  $ab^{2/d} > (\frac{d}{a+d})^2$ ,<sup>30</sup> then both equilibrium networks have a giant component.*

In some cases, both the low and the high equilibrium random graphs have a giant component. In some other cases, the low equilibrium random graph is fragmented while the high equilibrium contain a connected component encompassing a nontrivial fraction of all the players with high probability. In principle, Proposition 9 does not exclude the possibility of coexistence of two fragmented equilibrium networks (for high and low actions). We conjecture, however, that the only two possibilities are those described in the Proposition 9.

Holding  $b = \frac{1}{2}$ , Figure 3 displays for values of  $d$  ranging from 1 to 10,  $a_{\min}$  and  $a_{\max}$ , respectively the minimum and maximum values of  $a$  for which equilibrium exists.

The line in between  $a_{\min}$  and  $a_{\max}$  represents a phase transition separating two parameter regions for  $a$  and  $d$ . For values of  $a$  above this line, the two equilibrium random graphs have a

<sup>29</sup>See Vega-Redondo (2007) and Jackson (2008) for more details about the topological properties of Erdős-Rényi random graphs. Some of these topological properties carry over to the multinomial generalization of this random graph model, where links are independent Bernoulli random variable with heterogeneous success probabilities.

<sup>30</sup>Notice that  $(\frac{d}{a+d})^2 \geq (1 - a)^{2/d}$ . See the proof for details.

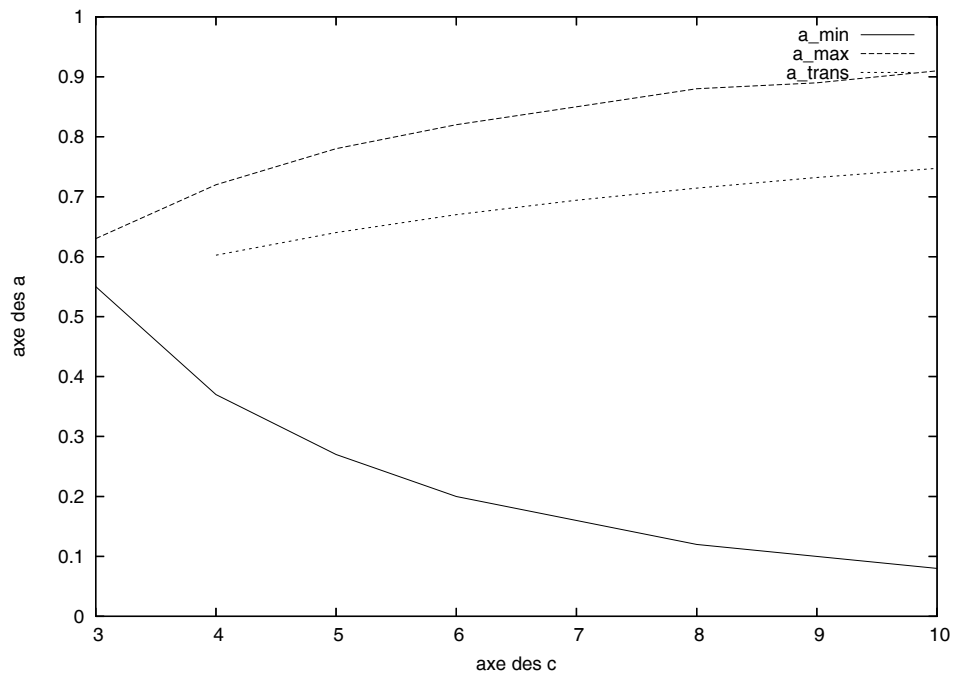


Figure 3:  $a_{\min}$  and  $a_{\max}$ ,

giant component. Instead, for values of  $a$  below this line, the low equilibrium random graph is fragmented.

Notice that this transition is sharp, that is, the topological properties of the low equilibrium change discontinuously as a function of the synergistic parameter  $a$ .

## 8 Discussion

### 8.1 Summary

We have provided a simple operational model of network formation with welfare and topology predictions, and clear-cut comparative statics. In substance, we identify a “too cold” and a “too hot” equilibrium. We show that socialization is more responsive than production to exogenous shocks in the parameters and that individual preferences over group composition leads to assortative community formation.

### 8.2 The literature on network formation and social interactions

A growing literature over the last two decades has examined the problem of social network formation and its implications for economic phenomena. With respect to this literature (see the survey of Jackson 2005) our model first fixes the boundaries of socialization but then socialization is modeled via a generic effort variable, rather than viewing network formation as the result of an earmarked socialization process (see also Jackson 2008).

In some cases, our model with synergies of varying strengths across different pairs of players<sup>31</sup> can also be understood as a (multinomial) model of a random graph, where independent link probabilities correspond to the synergistic values. Albeit simple, this random graph model is flexible enough to encompass many (if not all) of the topological properties of real life networks.<sup>32</sup> Our equilibrium analysis naturally inherits all the descriptive topological features of this random graph model (as well as its limitations, of course). At the same time, our analysis permits to draw a close connection between topological features and welfare and economic implications. In spite of its parsimony that ensures tractability, the model retains enough richness to replicate a (relatively) broad range of empirical regularities displayed by social and economic networks, and is directly estimable to recover its structural parameters.<sup>33</sup>

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<sup>31</sup>See Bloch and Dutta (2005) for a model with endogenous link strength but in a standard framework of non-random network formation.

<sup>32</sup>See Chung and Lu (2002) for an analysis of this random graph model, and Bender and Canfield (1978) for the analysis of the closely related configuration model. For an economic model of dynamic network formation leading to a broad class of random graph distributions, see Jackson and Rogers (2007).

<sup>33</sup>See Ioannides and Soetevent (2006) for an interesting application of random networks to the labor market.

### 8.3 The payoff function

We show and state our results with a class of payoffs corresponding to the functional form given in equation (4). However, the thrust of our analysis carries over to some generalizations of this setup.

The three main characteristics of this functional form are:

- (a) the linear-quadratic returns to productive investment,
- (b) the generic socialization effort (condition (A3) of Lemma 1),
- (c) aggregate constant returns to scale in socialization effort (condition (A2) of lemma 1).

We discuss possible generalizations of these three building-blocks.

Linear-quadratic production payoffs give rise to linear equilibrium conditions for the levels of productive investment. The results obtained with an homogeneous population and non-quadratic cost structures already shows that our analysis is robust to (at least some form of) non-linearities in these equilibrium conditions. Still, linear equilibrium conditions for productive investment play an important role in the analysis, as they allow to express existence and interiority of the productive investment decisions (for a given socialization profile  $\mathbf{s}$ ) as a function of the spectral radius of the matrix of link intensities  $\mathbf{G}(\mathbf{s}) = [g_{ij}(\mathbf{s})]$ . However, it is important to stress that linear-quadratic payoffs are not a *necessary* condition for equilibrium equations to be linear. Indeed, equilibrium analysis boils down to solving a so-called linear complementarity problem for a relatively broad class of payoff function, beyond the particular linear-quadratic specification. Ballester and Calvó-Armengol (2007) provide examples of these payoff functions. Our analysis would carry over to any such environment.

Combining the linear equilibrium equations for productive investment with conditions (A2) and (A3), Nash equilibrium conditions (both for productive investments and for socialization efforts) take a relatively manageable closed-form matrix expression. In turn, when the population gets large, and because we are able to control the population size effect in our matrix closed-form expression, approximate equilibrium conditions boil down to a simple system of equations (6).

On top of its operational virtues, condition (A3) is hard to dispense with as it embodies the central assumption of our approach, the genericity of socialization efforts, not earmarked to particular targets.

Condition (A2), instead, is chosen mainly for its operational virtues. We could accommodate variations of this condition, and thus alternative expressions for the link intensities  $g_{ij}(\mathbf{s})$  that allow for some aggregate scale effects in socialization, as long as population size effects can appropriately dealt with. Essentially, we need that both the spectral radius of  $\mathbf{G}(\mathbf{s})$  and that the diagonal cells of the matrix  $[\mathbf{I} - \mathbf{G}(\mathbf{s})]^{-1}$  are of finite order (with the population size), while the off-diagonal terms of the same matrix be of order inverse of the population size.

## 8.4 Multiplicity of equilibria

One slightly artificial feature of the model is the fact that the effort variables are unbounded. This creates existence problems and generates the need for the assumption  $2/3\sqrt{3} > a(\mathbf{b})$ . In addition, this generates a failure of upper-hemicontinuity in the equilibrium correspondence as the high-action equilibrium diverges to infinity as  $a(\mathbf{b})$  goes to zero. A simple way to deal with this problem is to assume that the effort of each individual is bounded. That is,  $s_i + k_i \leq T$ . This is natural when one interprets the sum of efforts of an individual as related to the time at his disposal, or, more generally as activities that consume resources of this sort. It is relatively easy to characterize the equilibria and their topological and welfare properties under this modification. In particular, the Pareto superior equilibrium disappears for  $a(\mathbf{b})T$  low enough. A bounded strategy space can introduce upper corner equilibria that may be stable.

Another economically compelling modification of the original setup is to introduce a market for effort resources that are limited in supply. For example, suppose that there is a fixed amount of productive effort  $\sum_{i=1}^n k_i = k$  sold in a competitive market. If one interprets (4) as the amount of numeraire produced by agent  $i$ , the total profit for an agent is:

$$\left( b_{\tau(i)} k_i + a \sum_{j=1}^n p_{ij}(\mathbf{s}) k_j k_i - \frac{1}{2} k_i^2 - \frac{1}{2} s_i^2 \right) - p_k k_i \quad (19)$$

which is equivalent to:

$$(b_{\tau(i)} - p_k) k_i + a \sum_{j=1}^n p_{ij}(\mathbf{s}) k_j k_i - \frac{1}{2} k_i^2 - \frac{1}{2} s_i^2 \quad (20)$$

This corresponds to our standard model, where the types are now  $b_{\tau(i)} - p_k$ . An equilibrium  $(p_k^*, \mathbf{k}^*, \mathbf{s}^*)$  is now simply a solution to (6) with the modified types and the market clearing conditions, for large populations.

It should be clear that, for a fixed (low enough)  $p_k$ , the new system (6) has a solution. For the low equilibrium, lowering types makes the  $k^*$  smaller coordinatewise (this is Proposition 2). This yields, in fact, a (downward sloping) demand function for  $k$ . Supply is a vertical line, so there is always an equilibrium (potentially at a price of zero and excess supply). The shape of the demand function for the high equilibrium is less straightforward because the comparative statics of the high equilibrium as  $p_k$  changes are now ambiguous (see again Proposition 2). In some cases, this demand function will actually be upward sloping at the high equilibrium. Because upward sloping demands give rise to unstable market dynamics, the low equilibrium is then uniquely selected.

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## Appendix

**Proof of Lemma 1:** Fix  $\mathbf{s}$ . Combining (A1) and (A3) gives  $s_k g_{ij}(\mathbf{s}) = s_j g_{ik}(\mathbf{s})$ . Summing across all  $j$ 's and using (A2) gives  $g_{ik}(\mathbf{s}) = s_i s_k / \sum_{j=1}^n s_j$ . ■

**Proof of Lemma 2:** The equilibrium analysis is clear. The stability result is established in Proposition 10 below. ■

**Proof of Theorem 1:** It follows from the following Lemmata 3, 4, 5 and 6. The stability analysis is in Proposition 10.

Consider an  $m$ -replica game involving  $n = mt$  players, where  $m \geq 1$  is fixed for the time being.

Let  $\mathbf{G}(\mathbf{s}) = [g_{ij}(\mathbf{s})]_{i,j \in N}$  be the  $n$ -symmetric adjacency matrix for the network with link intensities in (1).

For all  $\alpha \geq 0$  and for all  $\mathbf{x} \in \mathbb{R}_+^t$ , define  $\bar{\mathbf{x}} = \sum_{\tau=1}^t x_\tau / t$ ,  $\bar{\mathbf{x}}^2 = \sum_{\tau=1}^t x_\tau^2 / t$ , and:

$$\lambda_\alpha(\mathbf{x}) = \frac{\alpha \bar{\mathbf{x}}}{\bar{\mathbf{x}} - \alpha \bar{\mathbf{x}}^2}.$$

We extend this definition to any non-negative vector in an Euclidean space of arbitrary finite size.

**LEMMA 3** *Let  $\mathbf{s} \in \mathbb{R}_+^n$ ,  $\mathbf{s} \neq \mathbf{0}$  and  $\alpha \geq 0$  such that  $1 > \alpha \bar{\mathbf{s}}^2 / \bar{\mathbf{s}}$ . Then,  $\mathbf{M}(\mathbf{s}) = [\mathbf{I} - \alpha \mathbf{G}(\mathbf{s})]^{-1}$  is a well-defined and non-negative  $n$ -square matrix, equal to  $\mathbf{M}(\mathbf{s}) = \mathbf{I} + \lambda_\alpha(\mathbf{s}) \mathbf{G}(\mathbf{s})$ .*

**Proof.** When  $\mathbf{M}(\mathbf{s}) \in \mathbb{R}^{n^2}$  is well-defined, we have  $\mathbf{M}(\mathbf{s}) = \sum_{p=0}^{+\infty} \alpha^p \mathbf{G}(\mathbf{s})^p$ . We compute  $\mathbf{G}(\mathbf{s})^p$ . First, note that (we omit  $\mathbf{s}$  when there is no confusion):

$$g_{ij}^{[2]} = \sum_{l=1}^n g_{il} g_{lj} = \frac{s_i s_j}{n \bar{\mathbf{s}}} \sum_{h=1}^n \frac{s_h^2}{n \bar{\mathbf{s}}} = \frac{\bar{\mathbf{s}}^2}{\bar{\mathbf{s}}} g_{ij}, \text{ for all } i, j = 1, \dots, n$$

By a trivial induction on  $p = 1, 2, \dots$ , we deduce that  $g_{ij}^{[p]} = (\bar{\mathbf{s}}^2 / \bar{\mathbf{s}})^p g_{ij}$ , for all  $i, j$  and all  $p \geq 1$ . Therefore:

$$\mathbf{M}(\mathbf{s}) = \mathbf{I} + \sum_{p=1}^{+\infty} \left( \alpha \frac{\bar{\mathbf{s}}^2}{\bar{\mathbf{s}}} \right)^p \mathbf{G} = \mathbf{I} + \lambda(\mathbf{s}) \mathbf{G}(\mathbf{s}).$$

We know from Debreu and Herstein (1953) that  $\mathbf{M}(\mathbf{s})$  is well-defined and non-negative if and only if  $1 > \alpha \rho(\mathbf{G}(\mathbf{s}))$ , where  $\rho(\mathbf{G}(\mathbf{s}))$  is the modulus of the largest eigenvalue of  $\mathbf{G}(\mathbf{s})$  (see also Theorem 1 in Ballester, Calvó-Armengol and Zenou 2006). Let us show that  $\rho(\mathbf{G}(\mathbf{s})) = \bar{\mathbf{s}}^2 / \bar{\mathbf{s}}$ .

First, note that  $\bar{\mathbf{s}}^2 / \bar{\mathbf{s}}$  is an eigenvalue of  $\mathbf{G}$  for the eigenvector  $\mathbf{s}$ . Indeed,  $\mathbf{G} \cdot \mathbf{s} = (\bar{\mathbf{s}}^2 / \bar{\mathbf{s}}) \mathbf{s}$ .

Second, let  $\mathbf{x}$  such that  $\|\mathbf{x}\| = 1$ . We have:

$$\mathbf{G} \cdot \mathbf{x} = \frac{\mathbf{s} \cdot \mathbf{x}}{n \bar{\mathbf{s}}} \mathbf{s},$$

where  $\mathbf{s} \cdot \mathbf{x} = \sum_{i=1}^n s_i x_i$  is the scalar product, with  $|\mathbf{s} \cdot \mathbf{x}| \leq \|\mathbf{s}\| \times \|\mathbf{x}\| \leq \|\mathbf{s}\|$ . Therefore,  $\|\mathbf{G} \cdot \mathbf{x}\| \leq \|\mathbf{s}\|^2 / n \bar{\mathbf{s}} = \bar{\mathbf{s}}^2 / \bar{\mathbf{s}}$ . Note that, by definition,  $\rho(\mathbf{G}) = \sup \{ \|\mathbf{G} \cdot \mathbf{x}\| / \|\mathbf{x}\| : \|\mathbf{x}\| = 1 \}$ . Altogether, we can conclude that  $\rho(\mathbf{G}) = \bar{\mathbf{s}}^2 / \bar{\mathbf{s}}$ . ■

Let now  $m^1, m^2, m^3, \dots$  be an increasing sequence of integers such that  $m^h \rightarrow +\infty$  as  $h \rightarrow +\infty$ . Each  $h \in \mathbb{N}$  defines a  $m^h$ -replica game involving  $n^h = m^h t$  players. In the  $m^h$ -replica game, there are  $m^h$  players of each type  $(b_1, \dots, b_t)$ . In each such game, a profile of strategies is  $(\mathbf{k}^h, \mathbf{s}^h) \in \mathbb{R}_+^{n^h} \times \mathbb{R}_+^{n^h}$ . Given a player  $i = 1, \dots, n^h$ , recall that  $b_{\tau(i)}$  denotes his type, where  $\tau(i) \in T$ .

**LEMMA 4** *Let  $\{(\mathbf{s}^h, \mathbf{k}^h)\}_{h \in \mathbb{N}}$  be a sequence of Nash equilibria of the  $m^h$ -replica games such that  $c > a\overline{\mathbf{s}^{h2}}/\overline{\mathbf{s}^h}$ , for all  $h \in \mathbb{N}$ . Suppose that the system of equations:*

$$\begin{cases} [c - a(\mathbf{b})s]k = 1 \\ s = a(\mathbf{b})k^2 \end{cases} \quad (21)$$

has a solution  $(s, k) \in \mathbb{R}_+^2$  such that  $c > a(\mathbf{b})s$ . Then, for all  $\varepsilon > 0$ , there exists some  $h_\varepsilon \in \mathbb{N}$  such that, for all  $h \geq h_\varepsilon$ , we have  $\max\{|k_i^h - b_{\tau(i)}k|, |s_i^h - b_{\tau(i)}s|\} < \varepsilon$ , for all  $i = 1, \dots, n^h$ , where  $(s, k)$  is a solution to (21).

**Proof.** Let  $\{(\mathbf{k}^h, \mathbf{s}^h)\}_{h \in \mathbb{N}}$  be a sequence of Nash equilibria such that  $c > a(\mathbf{s}^h)$ , for all  $h$ . Let  $\text{diag}(\mathbf{G}(\mathbf{s}^h))$  be the diagonal matrix with diagonal terms  $g_{ii}(\mathbf{s}^h)$  and zero off-diagonal terms. For each  $h$ , the first-order necessary equilibrium conditions for  $k^h$  are:

$$[c\mathbf{I} - a\mathbf{G}(\mathbf{s}^h)] \cdot \mathbf{k}^h + a \text{diag}(\mathbf{G}(\mathbf{s}^h)) \cdot \mathbf{k}^h = \mathbf{b}^h.$$

Using the expression for  $[\mathbf{I} - \alpha\mathbf{G}(\mathbf{s}^h)]^{-1} \in \mathbb{R}^{n^{h2}}$  in Lemma 3 and letting  $\alpha = a/c$ , we rewrite this first-order necessary equilibrium conditions for  $k^h$  as:

$$c\mathbf{k}^h + a \left[ \mathbf{I} + \lambda_{a/c}(\mathbf{s}^h)\mathbf{G}(\mathbf{s}^h) \right] \cdot \text{diag}(\mathbf{G}(\mathbf{s}^h)) \cdot \mathbf{k}^h = \left[ \mathbf{I} + \lambda_{a/c}(\mathbf{s}^h)\mathbf{G}(\mathbf{s}^h) \right] \cdot \mathbf{b}^h, \quad (22)$$

where  $\mathbf{b}^h \in \mathbb{R}_+^{n^h}$  is defined by  $b_i^h = b_{\tau(i)}$ , for all  $i = 1, \dots, n^h$ . In words, the  $i$ th coordinate of  $\mathbf{b}^h$  corresponds to the private returns of player  $i$ 's type. Note that the  $n^h$  coordinates of  $\mathbf{b}^h$  take  $t$  different possible values,  $b_1, \dots, b_t$ , each repeated  $m^h$  times.

We also compute the first-order necessary equilibrium conditions for  $s_i^h$ , which are:

$$s_i^h = ak_i^h \frac{\mathbf{s}^h \cdot \mathbf{k}^h}{n^h \overline{\mathbf{s}^h}} - as_i^h k_i^h \frac{\mathbf{s}^h \cdot \mathbf{k}^h}{(n^h \overline{\mathbf{s}^h})^2} - a \frac{s_i^{h2} k_i^{h2}}{n^h \overline{\mathbf{s}^h}} + a \frac{s_i^{h2} k_i^{h2}}{(n^h \overline{\mathbf{s}^h})^2}, i = 1, \dots, n^h. \quad (23)$$

Given that  $c > a\overline{\mathbf{s}^{h2}}/\overline{\mathbf{s}^h}$ , for all  $h$  and that  $n^h \rightarrow +\infty$  as  $h \rightarrow +\infty$ , necessarily,  $s_i^h \in O(1)$ , for all  $i = 1, \dots, n^h$  and for all  $h$ . Indeed, suppose that  $s_i^h \in O(n^{hp})$ ,  $p > 0$ , for some  $j$ . Let then  $q > 0$  such that  $s_i^h \in O(n^{hq})$ ,  $q > 0$ , for all  $i$ . Then,  $a\overline{\mathbf{s}^{h2}}/\overline{\mathbf{s}^h} \in O(n^{h^q})$ , and the inequality  $c > a\overline{\mathbf{s}^{h2}}/\overline{\mathbf{s}^h}$  is violated for large enough  $h$ .

Given that  $s_i^h \in O(1)$ , we have  $g_{ij}(\mathbf{s}^h) = s_i^h s_j^h / \left( \sum_{l=1}^{n^h} s_l^h \right) \in o(1)$  when  $h \rightarrow +\infty$ , for all  $i, j = 1, \dots, n^h$

The first-order conditions (22) imply that  $k_i^h \in O(1)$ , for all  $i = 1, \dots, n^h$  and for all  $h$ .

Then, using (23), we deduce that for  $h$  high enough, we have

$$s_i^h = ak_i^h \frac{\mathbf{s}^h \cdot \mathbf{k}^h}{n^h \mathbf{s}^h} + o(1), \text{ for all } i = 1, \dots, n^h \text{ and for all } h.$$

By (22),  $k_i^h$  is a continuous function of  $\mathbf{s}^h$ . Therefore,  $s_i^h = \sigma_i^h + o(1)$  and  $k_i^h = \kappa_i^h + o(1)$ , for all  $i = 1, \dots, n^h$  and for all  $h$ , where  $(\sigma^h, \kappa^h)$  are such that:

$$c\kappa^h = \left[ \mathbf{I} + \lambda_{a/c}(\sigma^h) \mathbf{G}(\sigma^h) \right] \cdot \mathbf{b}^h, \quad (24)$$

and

$$\sigma_i^h = a\kappa_i^h \frac{\sigma_i^h \cdot \kappa_i^h}{n^h \sigma^h}, i = 1, \dots, n^h. \quad (25)$$

We solve (24) and (25).

Note, first, that (25) implies that  $\sigma_i^h / \kappa_i^h = \sigma_j^h / \kappa_j^h$ , for all  $i, j$ . Without any loss of generality, we can thus write  $\sigma_i^h = \theta_i^h s$  and  $\kappa_i^h = \theta_i^h k$ , for all  $i = 1, \dots, n^h$  and for some  $k, s$ . Then, (25) can be rewritten as:

$$s = ak^2 \frac{\overline{\theta^h}^2}{\theta^h}, \quad (26)$$

Note also that  $g_{ij}(\sigma^h) = \theta_i^h \theta_j^h s / n^h \overline{\theta^h}$  and that  $\lambda_{a/c}(\sigma^h) = a\overline{\theta^h} / (c\overline{\theta^h} - as\overline{\theta^h}^2)$ . We thus rewrite (24) as:

$$c\theta_i^h k = b_i^h + \frac{as}{n^h} \frac{\theta_i^h}{c\overline{\theta^h} - as\overline{\theta^h}^2} \sum_{j=1}^{n^h} \theta_j^h b_j^h, i = 1, \dots, n^h. \quad (27)$$

Let  $\theta_i^h = b_i^h$ . Then, (27) becomes:

$$ck = 1 + \frac{as\overline{\mathbf{b}}^2}{c\overline{\mathbf{b}} - as\overline{\mathbf{b}}^2} = \frac{c}{c - a(\mathbf{b})s},$$

equivalent to:

$$k = \frac{1}{c - a(\mathbf{b})s}$$

while (26) becomes  $s = a(\mathbf{b})k^2$ .

Finally, note that the condition  $c > a\overline{\mathbf{s}^h \mathbf{h}^2} / \overline{\mathbf{s}^h}$  is then equivalent to  $c > a(\mathbf{b})s$ . ■

**LEMMA 5** *If  $2(c/3)^{3/2} > a(\mathbf{b})$ , then the system of equations (21) has exactly two solutions  $(s, k) \in \mathbb{R}_+^2$  such that  $c > a(\mathbf{b})s$ .*

**Proof.** Plugging the expression for  $k$  into the expression for  $s$  in (21), one concludes that every solution  $(s^*, k^*)$  of (21) is such that  $g(s^*) = s^*$  where:

$$g(s) = \frac{a(\mathbf{b})}{(c - a(\mathbf{b})s)^2}. \quad (28)$$

We establish conditions such that the graph of  $g(s)$  crosses (twice) the 45 degree line for some  $s$  such that  $c > a(\mathbf{b})s$ . Note that  $g(0) = a(\mathbf{b})/c^2$  and  $\lim_{s \uparrow c/a(\mathbf{b})} g(s) = +\infty$ , so that the function  $g(\cdot)$  maps  $[0, c/a(\mathbf{b}))$  into  $[a(\mathbf{b})/c^2, +\infty)$ . We have:

$$g'(s) = \frac{2a(\mathbf{b})^2}{(c - a(\mathbf{b})s)^3}.$$

Therefore,  $g'(0) = 2a(\mathbf{b})^2/c^3$  and  $\lim_{s \uparrow c/a(\mathbf{b})} g'(s) = +\infty$ . If there exists a tangent to the graph of  $g(\cdot)$  on  $[0, c/a(\mathbf{b}))$  which is parallel to the 45 degree line, and if this tangent is strictly below (resp. tangent to) the 45 degree line, the system (21) has exactly two solutions (resp. one solution) on  $[0, c/a(\mathbf{b}))$ . Such a tangent exists if  $2a(\mathbf{b})^2 \leq c^3$ , equivalent to  $a(\mathbf{b}) \leq 2^{-1/2}c^{3/2}$ .

Next, we solve:

$$g'(x^*) = 1 \Leftrightarrow a(\mathbf{b})x^* = c - 2^{1/3}a(\mathbf{b})^{2/3} \quad (29)$$

Thus, (21) has two solutions (resp. one solution) if and only if  $a(\mathbf{b}) \leq 2^{-1/2}c^{3/2}$  and  $g(x^*) < x^*$  (resp.  $g(x^*) = x^*$ ), where  $x^*$  is defined by (29). We have:

$$g(x^*) = \frac{a(\mathbf{b})}{(2a(\mathbf{b})^2)^{2/3}} = \frac{a(\mathbf{b})^{-1/3}}{2^{2/3}}.$$

Therefore,

$$g(x^*) < x^* \Leftrightarrow a(\mathbf{b}) < 2c^{3/2}/3\sqrt{3}.$$

The overall condition is thus  $a(\mathbf{b}) < c^{3/2} \min\{2/3\sqrt{3}, 2^{-1/2}\}$ . However, it is readily checked that  $2/3\sqrt{3} < 2^{-1/2}$  (indeed, this is equivalent to  $2\sqrt{2} < 3\sqrt{3}$ ). When  $a(\mathbf{b}) < 2(c/3)^{3/2}$  (resp.  $a(\mathbf{b}) = 2(c/3)^{3/2}$ ), the graph of  $g(\cdot)$  thus crosses the 45 degree line twice (resp. once) on  $[0, c/a(\mathbf{b}))$ . ■

**LEMMA 6** *Let  $\{(s^{h*}, \mathbf{k}^{h*})\}_{h \in \mathbb{N}}$  be such that  $s_i^{h*} = sb_{\tau(i)}$  and  $k_i^{h*} = kb_{\tau(i)}$ , for all  $i = 1, \dots, n^h$ , where  $(s, k)$  is some given solution to (21). If  $2(c/3)^{3/2} > a(\mathbf{b})$ , then there exists some  $\bar{h} \in \mathbb{N}$  such that, for all  $h \geq \bar{h}$ , the second-order equilibrium conditions for  $\mathbf{u} = (u_1, \dots, u_{n^h})$  hold at  $(\mathbf{s}^{h*}, \mathbf{k}^{h*})$ .*

**Proof.** First note that Lemma 5 implies that (21) has a solution such that  $c > a(\mathbf{b})k$ . Consider this solution. We also know from Lemma 4 that both  $k_i^h, s_i^h \in O(1)$ , for all  $i = 1, \dots, n^h$ . We now compute the cross partial derivatives of  $\mathbf{u}$  at  $(\mathbf{s}^{h*}, \mathbf{k}^{h*})$ .

First, we have:

$$\frac{\partial u_i}{\partial s_i}(\mathbf{s}^{h*}, \mathbf{k}^{h*}) = \sum_{j=1, j \neq i}^{n^h} \left( \frac{a s_j^{h*} k_i^{h*} k_j^{h*}}{n^h \mathbf{s}^{h*}} - \frac{a g_{ij}(\mathbf{s}^{h*}) k_i^{h*} k_j^{h*}}{n^h \mathbf{s}^{h*}} \right) - s_i^{h*} \quad (30)$$

$$\frac{\partial u_i}{\partial k_i}(\mathbf{s}^{h*}, \mathbf{k}^{h*}) = b_{\tau(i)} + a \sum_{j=1, j \neq i}^{n^h} g_{ij}(\mathbf{s}^{h*}) k_j^{h*} - c k_i^{h*} \quad (31)$$

Thus

$$\begin{aligned}\frac{\partial^2 u_i}{\partial s_i^2}(\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) &= \sum_{j=1, j \neq i}^{n^h} \left( -\frac{as_j^{h^*} k_i^{h^*} k_j^{h^*}}{(n^h \overline{\mathbf{s}^{h^*}})^2} - \frac{as_j^{h^*} k_i^{h^*} k_j^{h^*}}{(n^h \overline{\mathbf{s}^{h^*}})^2} + \frac{2ag_{ij}(\mathbf{s}^{h^*}) k_i^{h^*} k_j^{h^*}}{(n^h \overline{\mathbf{s}^{h^*}})^2} \right) - 1 \\ \frac{\partial^2 u_i}{\partial s_i \partial k_i}(\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) &= \sum_{j=1, j \neq i}^{n^h} \left( \frac{as_j^{h^*} k_j^{h^*}}{n^h \overline{\mathbf{s}^{h^*}}} - \frac{ag_{ij}(\mathbf{s}^{h^*}) k_j^{h^*}}{n^h \overline{\mathbf{s}^{h^*}}} \right) \\ \frac{\partial^2 u_i}{\partial k_i^2}(\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) &= -c\end{aligned}$$

So, for  $h$  large enough, we get:

$$\frac{\partial^2 u_i}{\partial s_i^2}(\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) = o(1) - 1 \quad (32)$$

$$\frac{\partial^2 u_i}{\partial s_i \partial k_i}(\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) = o(1) + ak \frac{\overline{\mathbf{b}^2}}{\mathbf{b}} = o(1) + a(\mathbf{b})k \quad (33)$$

$$\frac{\partial^2 u_i}{\partial k_i^2}(\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) = -c \quad (34)$$

The second-order conditions amount to checking that the principal minors of the Hessian matrix

$$\begin{bmatrix} \frac{\partial^2 u_i}{\partial s_i^2} & \frac{\partial^2 u_i}{\partial s_i \partial k_i} \\ \frac{\partial^2 u_i}{\partial s_i \partial k_i} & \frac{\partial^2 u_i}{\partial k_i^2} \end{bmatrix} = \begin{bmatrix} o(1) - 1 & o(1) + a(\mathbf{b})k \\ o(1) + a(\mathbf{b})k & -c \end{bmatrix}$$

have alternating signs.

But the determinant of a matrix is a continuous (polynomial) function of the matrix entries. Given that (32) and (34), are negative, when  $h \rightarrow +\infty$ , we are thus left to check that the sign of the determinant is positive, that is,  $c - a(\mathbf{b})^2 k^2 > 0$ , which is equivalent to  $c - a(\mathbf{b})s > 0$ , which Lemma 5 shows is true when  $2(c/3)^{3/2} > a(\mathbf{b})$ . ■

**PROPOSITION 10** *For  $m$  sufficiently large, the two interior equilibria are stable while the equilibrium with  $(s_i^*, k_i^*) = (0, b_{\tau(i)}/c)$  for all  $i = 1, \dots, mt$  is not stable.*

**Proof:** Let  $\epsilon > 0$ . Take  $h$  large enough such that Theorem 1 holds for this  $\epsilon$ . We check stability by looking at the behavior of the gradient system

$$\frac{\partial k_i(t)}{\partial t} = \frac{\partial u_i(\mathbf{s}(t), \mathbf{k}(t))}{\partial k_i(t)} \quad (35)$$

$$\frac{\partial s_i(t)}{\partial t} = \frac{\partial u_i(\mathbf{s}(t), \mathbf{k}(t))}{\partial s_i(t)} \quad (36)$$

around the equilibrium points.

Let us first look at the partially corner equilibrium. By (30) we have that the first derivative with respect to  $k_i$  when  $h$  is large is

$$o(1) + \sum_{j=1, j \neq i}^{n^h} \frac{as_j^{h^*} k_i^{h^*} k_j^{h^*}}{n^h \mathbf{s}^{h^*}} - s_i^{h^*}$$

Consider a perturbation around the equilibrium  $s^\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ , with  $\underline{b} = \min\{b_1, \dots, b_n\}$ . Then, the first derivative with respect to  $s_i$  is approximately

$$ab_i \frac{\sum_{j=1}^n \varepsilon_j b_j}{\sum_{j=1}^n \varepsilon_j} - \varepsilon_i > ab_i \underline{b} - \varepsilon_i > 0,$$

for  $\varepsilon_i$  small enough. For any small enough perturbation,  $s_i$  would tend to increase for all  $i$ , thus negating stability.

If we linearize the dynamic system (35, 36) around the equilibria we get, for all  $i = 1, \dots, n^h$ :

$$\begin{aligned} \frac{\partial k_i(t)}{\partial t} &= \sum_{j=1}^{n^h} \frac{\partial^2 u_i}{\partial k_i \partial k_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) (k_j(t) - k_j^*) + \sum_{j=1}^{n^h} \frac{\partial^2 u_i}{\partial k_i \partial s_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) (s_j(t) - s_j^*) \quad (37) \\ \frac{\partial s_i(t)}{\partial t} &= \sum_{j=1}^{n^h} \frac{\partial^2 u_i}{\partial s_i \partial s_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) (s_j(t) - s_j^*) + \sum_{j=1}^{n^h} \frac{\partial^2 u_i}{\partial s_i \partial k_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) (k_j(t) - k_j^*) \end{aligned}$$

For  $i \neq j$ , we have:

$$\begin{aligned} \frac{\partial^2 u_i}{\partial s_i \partial s_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) &= \frac{ak_i^{h^*} k_j^{h^*}}{n^h \mathbf{s}^{h^*}} - \frac{as_i^{h^*} k_i^{h^*} k_j^{h^*}}{n^h \mathbf{s}^{h^*}} + \sum_{l=1, l \neq i}^{n^h} \frac{ag_{il} (\mathbf{s}^{h^*}) k_l^{h^*} k_l^{h^*}}{n^h \mathbf{s}^{h^*}} \\ \frac{\partial^2 u_i}{\partial s_i \partial k_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) &= \frac{as_j^{h^*} k_i^{h^*}}{n^h \mathbf{s}^{h^*}} - \sum_{l=1, l \neq j}^{n^h} \frac{ag_{jl} (\mathbf{s}^{h^*}) k_l^{h^*}}{n^h \mathbf{s}^{h^*}} \\ \frac{\partial^2 u_i}{\partial k_i \partial s_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) &= 0 \\ \frac{\partial^2 u_i}{\partial k_i \partial k_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) &= ag_{ij} (\mathbf{s}^{h^*}) \end{aligned}$$

Thus, we have when  $h$  gets large and for  $i \neq j$ :

$$\frac{\partial^2 u_i}{\partial s_i \partial s_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) = o(1) \quad (38)$$

$$\frac{\partial^2 u_i}{\partial s_i \partial k_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) = o(1) \quad (39)$$

$$\frac{\partial^2 u_i}{\partial k_i \partial s_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) = 0 \quad (40)$$

$$\frac{\partial^2 u_i}{\partial k_i \partial k_j} (\mathbf{s}^{h^*}, \mathbf{k}^{h^*}) = o(1) \quad (41)$$

The coefficients of the linearized gradient system (37) correspond to the cells of a  $2n^h \times 2n^h$  matrix  $\mathbf{\Pi}^h ((\mathbf{s}^{h*}, \mathbf{k}^{h*}))$  which, when  $h$  is large enough, gets arbitrarily close to the following matrix  $\mathbf{\Pi}^h$ :

$$\mathbf{\Pi}^h = \begin{bmatrix} \mathbf{A}, \mathbf{B} \\ \mathbf{B}, c\mathbf{A} \end{bmatrix}$$

where  $\mathbf{A}, \mathbf{B}$  are the following  $n^h \times n^h$  matrices

$$\mathbf{A} = \begin{bmatrix} -1, \dots, 0 \\ 0, \dots, -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} a(\mathbf{b})k, \dots, 0 \\ 0, \dots, a(\mathbf{b})k \end{bmatrix},$$

Let  $\lambda$  be an eigenvalue of  $\mathbf{\Pi}^h$  for some eigenvector  $\nu_\lambda^\top = [\mathbf{x}_\lambda^{h\top}, \mathbf{y}_\lambda^{h\top}]$ ,  $i = 1, \dots, n^h$  where  $\mathbf{x}_\lambda^h$  and  $\mathbf{y}_\lambda^h$  are  $n^h \times 1$  vectors with coordinates  $x_{i,\lambda}^h$  and  $y_{i,\lambda}^h$ ,  $i = 1, \dots, n^h$ . The eigenvalue and the eigenvector satisfy the following identity:  $\mathbf{\Pi}^h \nu_\lambda = \lambda \nu_\lambda$ , that is:

$$\begin{aligned} (\lambda + 1) x_{i,\lambda}^h - a(\mathbf{b})k y_{i,\lambda}^h &= 0, i = 1, \dots, n^h \\ -a(\mathbf{b})k x_{i,\lambda}^h + (\lambda + c) y_{i,\lambda}^h &= 0, i = 1, \dots, n^h \end{aligned}$$

These systems have a non-trivial solution if and only if  $(\lambda + 1)(\lambda + c) - (a(\mathbf{b})k)^2 = 0$ . This second-order equation has a discriminant equal to  $(c - 1)^2 + 4(a(\mathbf{b})k)^2 \geq 0$ . It thus always admits two real solutions, denoted  $\lambda^1$  and  $\lambda^2$ . Besides:

$$\lambda^1 + \lambda^2 = -(1 + c) < 0,$$

implying that at least one such solution is negative. Then, both are negative if and only if  $\lambda^1 \lambda^2 = c - (a(\mathbf{b})k)^2 > 0$ , equivalent to  $c - a(\mathbf{b})k > 0$ , which is true when  $2(c/3)^{3/2} > a(\mathbf{b})$ .

Summing up, when  $2(c/3)^{3/2} > a(\mathbf{b}) > 0$ , the matrix  $\mathbf{\Pi}^h$  has two different negative eigenvalues  $\lambda^1, \lambda^2 < 0$ . Each eigenvalue  $\lambda^r$ ,  $r = 1, 2$  has an associated eigenspace of dimension  $n^h$  with generating eigenvectors  $[a(\mathbf{b})k \mathbf{i}^{h\top}, (\lambda^r + 1) \mathbf{i}^{h\top}]$ ,  $i = 1, \dots, n^h$  where  $\mathbf{i}^h$  is an  $n^h \times 1$  vector containing a 1 in position  $i = 1, \dots, n^h$  and 0's in the other  $n^h - 1$  positions. ■

**Proof of Proposition 1:** Let  $\epsilon > 0$ . Take  $h$  large enough such that Theorem 1 holds for this  $\epsilon$ , and let  $(\mathbf{s}^{h*}, \mathbf{k}^{h*})$  and  $(\mathbf{s}^{h**}, \mathbf{k}^{h**})$  be the two approximate equilibria, where  $s_i^{h*} = s^* b_{\tau(i)}$ ,  $k_i^{h*} = k^* b_{\tau(i)}$ , and  $s_i^{h**} = s^{**} b_{\tau(i)}$ ,  $k_i^{h**} = k^{**} b_{\tau(i)}$ , for all  $i = 1, \dots, n^h$ , and  $(s^*, k^*), (s^{**}, k^{**})$  are the two different solutions to (6). Suppose that  $s_i^{h*} \geq s_i^{h**}$ , for some  $i$ . Then, necessarily,  $s^* \geq s^{**}$ . By (6), we deduce that  $k^* \geq k^{**}$ . Therefore, both  $s_i^{h*} \geq s_i^{h**}$  and  $k_i^{h*} \geq k_i^{h**}$ , for all  $i = 1, \dots, n^h$ .

To establish the welfare ranking, we first use the expression for payoffs in (4) and the first-order conditions for  $k_i^h$ , to obtain the following expression for approximate equilibrium payoffs:

$$u_i^{h*} = \frac{b_{\tau(i)}^2}{2} (ck^{*2} - s^{*2}) + o(1), \text{ for all } i = 1, \dots, n^h.$$

Next, using the fact that  $(s^*, k^*)$  are solutions to (6), we write:

$$ck^{*2} - s^{*2} = c \frac{s^*}{a(\mathbf{b})} - s^{*2} = \frac{s^*}{a(\mathbf{b})} (c - a(\mathbf{b})s^*) = \frac{1}{a(\mathbf{b})} \frac{s^*}{k^*} = k^*, \quad (42)$$

and thus:

$$u_i^{h*} = \frac{b_{\tau(i)}^2}{2} k^* + o(1), \text{ for all } i = 1, \dots, n^h, \quad (43)$$

and similarly for the approximate equilibrium payoffs  $u_i^{h**}$  that correspond to  $(\mathbf{s}^{h**}, \mathbf{k}^{h**})$ . Since, by definition  $k^* \geq k^{**}$ , the welfare at the equilibrium  $(\mathbf{s}^{h*}, \mathbf{k}^{h*})$  is higher than at the equilibrium  $(\mathbf{s}^{h**}, \mathbf{k}^{h**})$ .

To establish the position of the efficient outcome, the planners' problem is:

$$\sum_{i=1}^n \left( b_{\tau(i)} k_i + a \sum_{j=1, j \neq i}^n g_{ij}(\mathbf{s}) k_j k_i - \frac{1}{2} c k_i^2 - \frac{1}{2} s_i^2 \right)$$

for which FOC's are given by:

$$ck^h + 2a \left[ \mathbf{I} + \lambda_{a/c}(\mathbf{s}^h) \mathbf{G}(\mathbf{s}^h) \right] \cdot \text{diag}(\mathbf{G}(\mathbf{s}^h)) \cdot \mathbf{k}^h = \left[ \mathbf{I} + \lambda_{a/c}(\mathbf{s}^h) \mathbf{G}(\mathbf{s}^h) \right] \cdot \mathbf{b}^h, \quad (44)$$

$$s_i^h = 2ak_i^h \frac{\mathbf{s}^h \cdot \mathbf{k}^h}{n^h \bar{\mathbf{s}}^h} - 2as_i^h k_i^h \frac{\mathbf{s}^h \cdot \mathbf{k}^h}{(n^h \bar{\mathbf{s}}^h)^2} - 2a \frac{s_i^h k_i^{h2}}{n^h \bar{\mathbf{s}}^h} + 2a \frac{s_i^{h2} k_i^{h2}}{(n^h \bar{\mathbf{s}}^h)^2}, i = 1, \dots, n^h. \quad (45)$$

Notice that conditions (44) and (45) are like (23) and (22) except with  $2a$  rather than  $a$ , and therefore its approximate version is:

$$\begin{cases} s = 2a(\mathbf{b})k^2 \\ k [c - 2a(\mathbf{b})s] = 1 \end{cases} \quad (46)$$

We can thus use the results of proposition elasticity to establish that the (two) solutions of (46) will lie between the two solution of 6. Since the second order conditions will be satisfied (by a reason analogous to the one in lemma (6), one of those solutions will be the optimal point. Indeed, by the reasoning in this same proposition the componentwise higher solution is the optimal one. The result now follows. ■

**Proof of Propositions 2:** Let  $\epsilon > 0$ . Take  $h$  large enough such that Theorem 1 holds for this  $\epsilon$ . We denote by  $(\mathbf{s}^{h*}, \mathbf{k}^{h*})$  and  $(\mathbf{s}^{h**}, \mathbf{k}^{h**})$  the corresponding  $\epsilon$ -equilibria, where  $(\mathbf{s}^{h*}, \mathbf{k}^{h*}) \geq (\mathbf{s}^{h**}, \mathbf{k}^{h**})$  are computed with two different solutions  $(s^*, k^*) \geq (s^{**}, k^{**})$  of (21). On the  $(s, k)$  plane, an increase in  $a(\mathbf{b})$  results in an upward shift of the graph of:

$$k = \frac{1}{c - a(\mathbf{b})s}, \quad (47)$$

and a downward shift of the graph of:

$$s = a(\mathbf{b})k^2. \quad (48)$$

Therefore, the equilibrium actions of the Pareto-inferior equilibrium  $(\mathbf{s}^{h**}, \mathbf{k}^{h**})$  all increase, while those of the Pareto-superior equilibrium  $(\mathbf{s}^{h*}, \mathbf{k}^{h*})$  all decrease. The elasticity  $\eta$  that keeps track of the relative changes on  $s$  and  $k$  when  $a(\mathbf{b})$  varies is:

$$\eta = \frac{s}{k} \frac{\frac{\partial k}{\partial a(\mathbf{b})}}{\frac{\partial s}{\partial a(\mathbf{b})}}.$$

Differentiating (47) and (48) with respect to  $a(\mathbf{b})$  gives:

$$\begin{aligned} \frac{\partial k}{\partial a(\mathbf{b})} &= sk^2 + a(\mathbf{b})k^2 \frac{\partial s}{\partial a(\mathbf{b})} \\ \frac{\partial s}{\partial a(\mathbf{b})} &= k^2 + 2\frac{s}{k} \frac{\partial k}{\partial a(\mathbf{b})} \end{aligned}$$

Solving for the two partial derivatives gives:

$$\frac{\partial k}{\partial a(\mathbf{b})} = \frac{2sk^2}{1 - 2a(\mathbf{b})sk} \quad (49)$$

$$\frac{\partial s}{\partial a(\mathbf{b})} = \frac{k^2}{1 - 2a(\mathbf{b})sk} \left( 2\frac{s^2}{k} + 1 \right) \quad (50)$$

and thus:

$$\eta = \frac{s}{k} \frac{\frac{\partial k}{\partial a(\mathbf{b})}}{\frac{\partial s}{\partial a(\mathbf{b})}} = \frac{2s^2}{k + 2s^2} < 1.$$

■

**Proof of Proposition 4:** It is readily checked that  $\partial a(\mathbf{b})/\partial b_\tau > 0$  if and only if  $b_\tau^2 + 2b_\tau \sum_{\tau' \neq \tau} b_{\tau'} \geq 2 \sum_{\tau' \neq \tau} b_{\tau'}^2$ , for all  $\tau \in T$ , a sufficient condition for which is  $2b_1 \geq b_t$ . ■

**Proof of Proposition 5:** Let  $\epsilon > 0$ . Take  $h$  large enough such that Theorem 1 holds for this  $\epsilon$ . We denote by  $(\mathbf{s}^{h*}, \mathbf{k}^{h*})$  and  $(\mathbf{s}^{h**}, \mathbf{k}^{h**})$  the corresponding  $\epsilon$ -equilibria, where  $(\mathbf{s}^{h*}, \mathbf{k}^{h*}) \geq (\mathbf{s}^{h**}, \mathbf{k}^{h**})$  are computed with two different solutions  $(s^*, k^*) \geq (s^{**}, k^{**})$  of (6).

Using (43) and (6), and summing across all players gives the following expression for the population average equilibrium payoffs:

$$\sum_{i=1}^{n^h} u_i^{h*} = \frac{1}{n^h} \sum_{i=1}^{n^h} \frac{b_{\tau(i)}^2}{2a(\mathbf{b})} \frac{s^*}{k^*} + o(1) = \frac{1}{at} \frac{s^*}{k^*} \sum_{\tau=1}^t b_\tau + o(1).$$

Following a mean preserving spread in individual traits,  $a(\mathbf{b})$  increases and we can conclude. ■

**Proof of Proposition 6:** We first prove that top-down sorting yields to an unambiguous ranking of the compound index across any two communities.

Take two communities  $\ell > \ell'$ . Denote by  $\mathbf{b}^\ell, \mathbf{b}^{\ell'}$  the type profiles at these locations under a top-down sorting assignment, and by  $T^\ell, T^{\ell'} \subset T$  the subset of types present at these communities. We have:

$$a(\mathbf{b}^\ell) = a \frac{\sum_{\tau \in T^\ell} m_\tau b_\tau^2}{\sum_{\tau \in T^\ell} m_\tau b_\tau},$$

where  $m_\tau$  is the number of players of type  $\tau$  in community  $\ell$ . The compound index  $a(\mathbf{b}^{\ell'})$  for community  $\ell'$  is defined similarly.

Then,  $a(\mathbf{b}^\ell) \geq a(\mathbf{b}^{\ell'})$  is equivalent to:

$$\sum_{\tau \in T^\ell, \tau' \in T^{\ell'}} m_\tau m_{\tau'} b_\tau^2 b_{\tau'} \geq \sum_{\tau \in T^{\ell'}, \tau' \in T^{\ell'}} m_\tau m_{\tau'} b_\tau b_{\tau'}^2.$$

Recall that  $\ell > \ell'$ . Under top-down sorting,  $\tau \in T^\ell, \tau' \in T^{\ell'}$  implies that  $b_\tau \geq b_{\tau'}$ , with a strict inequality for at least one pair  $\tau, \tau'$  if  $m > n_\ell + n_{\ell+1}$  for all  $\ell = 1, \dots, L-1$ , that is, two successive communities cannot be fully occupied by players of the same type.

The argument is symmetric for bottom-up sorting, where  $\ell > \ell'$  implies  $a(\mathbf{b}^\ell) < a(\mathbf{b}^{\ell'})$ . Equilibrium sorting prices are then simply obtained by writing down moving incentives. ■

**Proof of Proposition 7:** We first show that the baseline productive investment is strictly convex (resp. strictly concave) in  $a(\mathbf{b})$  at the low (resp. high) equilibrium. Combining the two equations in (6), we obtain the following third-order polynomial in the baseline productive investment:  $a(\mathbf{b})^2 k^3 - kc + 1 = 0$ .

Differentiating twice gives:

$$\begin{cases} \frac{\partial k}{\partial a(\mathbf{b})} [c - 3a(\mathbf{b})^2 k^2] = 2a(\mathbf{b}) k^3 \\ \frac{\partial^2 k}{\partial a(\mathbf{b})^2} [c - 3a(\mathbf{b})^2 k^2] = 2k^3 + 12a(\mathbf{b}) k^2 \left( \frac{\partial k}{\partial a(\mathbf{b})} \right) + 6a(\mathbf{b})^2 k \left( \frac{\partial k}{\partial a(\mathbf{b})} \right)^2 \end{cases} .$$

We know from Proposition 2 that the the baseline productive investment is increasing (resp. decreasing) in  $a(\mathbf{b})$  at the low (resp. high) equilibrium. Therefore,  $c - 3a(\mathbf{b})^2 k^2$  takes a positive (resp. negative) sign at the low (resp. high) equilibrium. The expression for  $\partial^2 k / \partial a(\mathbf{b})^2$  then implies that this second-order derivative is positive at the low equilibrium.

We focus on the low equilibrium.

Consider two communities under top-down sorting,  $\ell > \ell'$ . We denote by  $\widehat{\ell}$  the community obtained by mixing communities  $\ell$  and  $\ell'$ . Straight algebra shows that:

$$a(\mathbf{b}^{\widehat{\ell}}) = (1 - \lambda) a(\mathbf{b}^{\ell'}) + \lambda a(\mathbf{b}^\ell),$$

where<sup>34</sup>

$$\lambda = \frac{a(\mathbf{b}^{\widehat{\ell}}) - a(\mathbf{b}^{\ell'})}{a(\mathbf{b}^{\ell}) - a(\mathbf{b}^{\ell'})} \in [0, 1].$$

Given the strict convexity of baseline production at the low equilibrium, we deduce that:

$$k^*(a(\mathbf{b}^{\widehat{\ell}})) < (1 - \lambda) k^*(a(\mathbf{b}^{\ell'})) + \lambda k^*(a(\mathbf{b}^{\ell})) \quad (51)$$

Total welfare for the top-down segregated communities  $\ell, \ell'$  is higher than total welfare for the mixed community  $\widehat{\ell}$  if and only if:

$$k^*(a(\mathbf{b}^{\widehat{\ell}})) \sum_{i \in \ell, \ell'} b_{\tau(i)}^2 < k^*(a(\mathbf{b}^{\ell'})) \sum_{i \in \ell'} b_{\tau(i)}^2 + k^*(a(\mathbf{b}^{\ell})) \sum_{i \in \ell} b_{\tau(i)}^2, \quad (52)$$

where, abusing slightly notation, we write  $i \in \widehat{\ell}$  to signify that player  $i$  belongs to community  $\widehat{\ell}$ .

Define:

$$\beta = \frac{\sum_{i \in \ell} b_{\tau(i)}^2}{\sum_{i \in \ell, \ell'} b_{\tau(i)}^2} \in [0, 1].$$

Then, we rewrite (52) as:

$$k^*(a(\mathbf{b}^{\widehat{\ell}})) < (1 - \beta) k^*(a(\mathbf{b}^{\ell'})) + \beta k^*(a(\mathbf{b}^{\ell})) \quad (53)$$

We know from the proof of Proposition 6 that  $a(\mathbf{b}^{\ell}) > a(\mathbf{b}^{\ell'})$ . Also, the baseline productive investment increases with the compound index  $a(\mathbf{b})$  at the low equilibrium. Therefore, given (51), a sufficient condition for (53) to hold is that  $\beta > \lambda$ , that is:

$$\frac{\sum_{i \in \ell} b_{\tau(i)}^2}{\sum_{i \in \ell, \ell'} b_{\tau(i)}^2} > \frac{a(\mathbf{b}^{\widehat{\ell}}) - a(\mathbf{b}^{\ell'})}{a(\mathbf{b}^{\ell}) - a(\mathbf{b}^{\ell'})},$$

which is equivalent to:

$$\left( \frac{\sum_{i \in \ell} b_{\tau(i)}^2}{\sum_{i \in \ell} b_{\tau(i)}} - \frac{\sum_{i \in \ell'} b_{\tau(i)}^2}{\sum_{i \in \ell'} b_{\tau(i)}} \right) \frac{\sum_{i \in \ell} b_{\tau(i)}}{\sum_{i \in \ell, \ell'} b_{\tau(i)}^2} > \frac{\sum_{i \in \ell, \ell'} b_{\tau(i)}^2}{\sum_{i \in \ell, \ell'} b_{\tau(i)}} - \frac{\sum_{i \in \ell'} b_{\tau(i)}^2}{\sum_{i \in \ell'} b_{\tau(i)}}.$$

Multiplying both sides by  $\sum_{i \in \ell} b_{\tau(i)} \sum_{i \in \ell'} b_{\tau(i)} \sum_{i \in \ell, \ell'} b_{\tau(i)} \sum_{i \in \ell, \ell'} b_{\tau(i)}^2$  leads to, after some algebra:

$$\left( \sum_{i \in \ell'} b_{\tau(i)} \sum_{i \in \ell} b_{\tau(i)}^2 - \sum_{i \in \ell} b_{\tau(i)} \sum_{i \in \ell'} b_{\tau(i)}^2 \right)^2 > 0,$$

which is true.

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<sup>34</sup>Notice that straight algebra establishes that  $a(\mathbf{b}^{\ell}) \geq a(\mathbf{b}^{\widehat{\ell}}) \geq a(\mathbf{b}^{\ell'})$ , where  $a(\mathbf{b}^{\ell}) > a(\mathbf{b}^{\ell'})$  under top-down sorting if community sizes are such that at least two different types are present in two contiguous communities, which we assume throughout.

Therefore, (53) holds true. Successive comparisons of social welfare for two segregated communities versus the merged community leads to the result on social welfare ranking. ■

**Proof of Theorem 2:** The proof follows *ceteris paribus* from that of Theorem 1.

First, we rewrite Lemma 4 by simply taking to the power of  $d$  the left-hand side terms in the first-order conditions (22) and (23), and the approximated first-order conditions (24) and (25). Then, (26) and (27) yields (16).

Second, rewriting equation (28) in Lemma 5 we get:

$$g_d(s) = \frac{a^{\frac{1}{d}} b^{\frac{2}{d^2}}}{1 - as^{\frac{2}{d^2}}},$$

so that the solution to (29) is such that  $(1 - ax^*)^{\frac{2}{d^2}+1} = \frac{2}{d^2} a^{1+\frac{1}{d}} b^{\frac{2}{d^2}}$ , that is,  $ax^* = 1 - \left(\frac{2}{d^2} a^{1+\frac{1}{d}} b^{\frac{2}{d^2}}\right)^{\frac{d^2}{2+d^2}}$ .

The equilibrium conditions  $g_d(x^*) < x^*$  then boils down to  $a^{1+\frac{1}{d}} b^{\frac{2}{d^2}} \leq \frac{d^2}{2+d^2} \left(\frac{2}{2+d^2}\right)^{\frac{2}{d^2}}$  after some simple algebra. ■

**Proof of Proposition 8:** Let's multiply both  $a$  and  $b$  by a common factor  $\delta \geq 1$ . Equations (16) become:

$$\begin{cases} s^d = \delta ak^2 \\ k^d (1 - \delta as) = \delta b \end{cases}.$$

Differentiating with respect to  $\delta$  and letting  $\delta = 1$  gives:

$$\begin{aligned} ds^{d-1} \frac{\partial s}{\partial \delta} \Big|_{\delta=1} &= ak^2 + 2ak \frac{\partial k}{\partial \delta} \Big|_{\delta=1} \\ dk^{d-1} \frac{\partial k}{\partial \delta} \Big|_{\delta=1} (1 - as) &= b + k^d \left[ as + a \frac{\partial s}{\partial \delta} \Big|_{\delta=1} \right] \end{aligned}$$

Straight algebra gives:

$$\begin{aligned} d \frac{\partial s}{\partial \delta} \Big|_{\delta=1} - 2 \frac{s}{k} \frac{\partial k}{\partial \delta} \Big|_{\delta=1} &= s \\ -ak \frac{\partial s}{\partial \delta} \Big|_{\delta=1} + d(1 - as) \frac{\partial k}{\partial \delta} \Big|_{\delta=1} &= k \end{aligned}$$

Finally, solving for the two partial derivatives yields to

$$\begin{aligned} \frac{\partial s}{\partial \delta} \Big|_{\delta=1} &= \frac{2 + d}{d^2(1 - as) - 2as} s \\ \frac{\partial k}{\partial \delta} \Big|_{\delta=1} &= \frac{as + d(1 - as)}{d^2(1 - as) - 2as} s \end{aligned} \tag{54}$$

Therefore, both partial derivatives are of the same sign, which is positive if and only if  $as < d^2/(2 + d^2)$ . In the same spirit of the proof of Proposition 5, one can check graphically that the

equilibrium actions of the low equilibrium increase with  $\delta$  while the equilibrium actions of the high equilibrium decrease with  $\delta$ . Then, noting from proposition (2) that the elasticity is smaller than one whenever the slope  $s/k$  increases,  $k$  and  $s$  either all increase or all decrease, the result follows. ■

**Proof of Proposition 9:** At an interior equilibrium, the strategies  $(s, k)$  solve:

$$\begin{cases} s^d = ak^2 \\ k^d(1 - as) = b \end{cases},$$

The first equation is:

$$s = f(k) = a^{1/d}k^{2/d}.$$

It is readily checked that  $f(\cdot)$  is increasing, strictly concave (resp. convex) when  $d < 2$  (resp.  $d > 2$ ) and a straight line when  $d = 2$  on  $[0, +\infty)$ . Also,  $f(0) = 0$  and  $\lim_{x \uparrow +\infty} f(x) = +\infty$ . Finally, note that:

$$k = f^{-1}(s) = \left(\frac{s}{a^{1/d}}\right)^{d/2}.$$

The second equation is:

$$k = g(s) = b^{1/d}(1 - as)^{-1/d}.$$

It is readily checked that  $g(\cdot)$  is increasing and strictly convex on  $[0, 1/a)$ , with  $g(0) = b^{1/d}$  and  $\lim_{x \uparrow 1/ab} g(x) = +\infty$ .

Suppose that the equilibrium existence conditions hold. Let  $(\underline{s}, \underline{k})$  and  $(\bar{s}, \bar{k})$  be the two different equilibria, where  $\bar{k} > \underline{k}$  (and, thus,  $\bar{s} > \underline{s}$ ). The two corresponding equilibrium networks are such that only one has a giant component if and only if  $\bar{s} > 1 > \underline{s}$ .

Suppose first that  $a \geq 1$ . Then,  $1 \geq 1/a$ , so that the threshold value 1 is to the right of the asymptote for  $g(\cdot)$ , implying that  $1 > \bar{s} > \underline{s}$ , that is, no equilibrium network has a giant component.

Suppose now that  $a < 1$ .

Note that the graph of  $g(\cdot)$  lies below the graph of  $f^{-1}(\cdot)$  only when  $s \in [\underline{s}, \bar{s}]$ . Therefore,  $\bar{s} > 1 > \underline{s}$  is equivalent to  $g(1) < f^{-1}(1)$ . After some algebra, this is equivalent to:

$$ab^{2/d} < (1 - a)^{2/d}. \tag{55}$$

Notice that the right-hand side of this inequality is well-defined given our assumption that  $a < 1$ .

When (55) holds, we can conclude that the two equilibrium networks display two different topological characteristics: the densely connected network has a giant component whereas the sparsely connected network doesn't.

Reciprocally, when (55) does not hold, we can conclude that either both equilibrium networks have a giant component, or none does. We now provide an additional sufficient condition such that both have giant component when both  $a < 1$  and  $ab^{2/d} > (1 - a)^{2/d}$ .

Consider the line tangent to the graph of  $g(\cdot)$  at the point  $(0, g(0))$  with equation:

$$h(s) = g'(0)s + g(0) = b^{1/d} \left[ \frac{a}{d}s + 1 \right].$$

The graph of  $g(\cdot)$  lies above that of  $h(\cdot)$  on  $[0, 1/a)$  (recall that  $g(\cdot)$  is strictly convex on that half-segment). Consider the region on the  $(s, k)$  space delimited to the left by the vertical axis, from above by the graph of  $g(\cdot)$ , from below by the graph of  $f^{-1}(\cdot)$  and to the right by the point  $(\underline{s}, \underline{k})$  at the intersection of these two graphs. If the point  $(1, h(1))$  lies in this region, then necessarily  $1 < \underline{s}$ . Analytically,  $(1, h(1))$  lies in this region if and only if  $h(1) > f^{-1}(1)$ . After some algebra, this is equivalent to:

$$ab^{2/d} > \left( \frac{d}{a+d} \right)^2.$$

When  $a < 1$ , we can thus conclude that both networks have a giant component when:

$$ab^{2/d} > \max\{(1-a)^{2/d}, \left( \frac{d}{a+d} \right)^2\}.$$

We now compare the two terms on the right-hand side of the inequality. Let:

$$\xi(x) = \frac{1}{1+ax} \quad \text{and} \quad \zeta(x) = (1-a)^x, \quad \text{where } x \in [0, 1].$$

Then,  $\left( \frac{d}{a+d} \right)^2 > (1-a)^{2/d}$  for some  $d \geq 1$  if and only if  $\xi(1/d) > \zeta(1/d)$ .

Note that  $\xi(0) = \zeta(0) = 1$ ,  $\xi(1) = 1/(1+a) \geq \zeta(1) = 1-a$  (with a strict inequality when  $a \neq 0$ ), and  $\xi'(0) = -a > \zeta'(0) = \log(1-a)$ . Given the strict convexity of both functions on  $[0, 1]$ , we can conclude that  $\xi(x) > \zeta(x)$  on  $0 < x < 1$ . ■