Do Auctions Select Efficient Firms?[†]

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Abstract. This paper considers a government auctioning off multiple licenses to firms that compete in a market after the auction. Firms have different costs, and cost efficiency is private information at the auction stage and at the market competition stage. If only one license is auctioned, standard results say that the most efficient firm wins the auction (license) as it will get the highest profit in the aftermarket, *i.e.*, it has the highest valuation for the license. This paper argues that this result does not generalize to the case of multiple licenses and aftermarket competition. In particular, we determine conditions under which auctions may select inefficient firms and therefore lead to an inefficient allocation of resources. Strategic interaction in the aftermarket, in particular the fact that firms prefer to compete with the least cost-efficient firms rather than with the most efficient firms, is responsible for this result.

Key Words: Auctions, cost-efficiency, aftermarkets

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1. Introduction

In many liberalization or privatization projects, governments eventually face the issue how to select firms that will provide the formerly publicly provided service. One of the advantages of using auctions as a selection mechanism, so it is often thought, is that auctions select the most cost-efficient firms. Markets where active firms are more cost-efficient typically yield more efficient market outcomes than when these same markets are served by less cost-efficient firms. Other things being equal, cost-efficiency seems to be good for overall economic welfare.

In a monopoly context, the most cost-efficient firm will win the competition for the market (read: will win the auction). In this paper, we refer to this result as the monopoly result. This result has permeated a large literature on procurement issues and, indeed, Laffont and Tirole (2002, pp. 307-8) state that if one ignores the processing, capture and dynamic costs of auctions, it is easy to see that auctions typically select the firm with the lowest cost. They attribute this argument to Demsetz (1968) who argued that competition for the market might be a good substitute for competition in the market.

Recently, many governments have relied on a combination of "competition for" and "competition in" the market, indicating that it is not just auction revenue that the government is interested in.[1] Competition in the market guarantees that the inefficiency of the market allocation is as small as possible. This is achieved by auction designs that stipulate that each winning firm can obtain one license.[2] Competition for the market guarantees that the government receives a financial return that is in line with their value. An important case in point is the wave of 3G mobile telephony spectrum auctions that have been organized around the world.[3] In all of the 3G auctions, multiple licenses were sold, and as typically, there were more firms participating in the auction than

[1] As monopoly profits are higher than total industry profits under oligopoly, governments may consider selling just one license if auction revenue were the prime objective. When costs are sufficiently low and nearly symmetric across firms, and asymmetries between players do not play a significant role (as they do in, *e.g.*, Hoppe et *al.*, 2006), the efficiency gains due to scale economies under monopoly are outweighed by the extra consumer surplus under oligopoly, making selling multiple licenses the optimal thing to do.

[2] In the upcoming 2.6 GHz auction in the UK and other European countries, the government usually imposes a maximum spectrum band to be obtained by one firm, again guaranteeing that multiple firms win spectrum.

[3] See Klemperer (2002a), (2002b), Binmore and Klemperer (2002) and Jehiel and Moldovanu (2003) for overviews of 3G mobile telephony spectrum auctions that have been held around the world.

available licenses, firms had to compete to obtain a license. Many governments formally or informally stated that efficient assignment of frequency spectrum was one of the goals to be achieved. With cost asymmetries between firms, efficient assignment implies that the most cost-efficient firms should win licenses, and indeed the Dutch government, among others, mentioned selecting the most efficient firms as one of the reasons for holding an auction (see, *e.g.*, Janssen *et al.*, 2001).

In this paper, we argue, however, that the monopoly result does *not* carry over to the case of multiple licenses, *i.e.*, to the case where firms compete in an oligopolistic fashion in the aftermarket. Due to strategic interactions between firms in the aftermarket, it is not necessarily true that at the auction stage the most cost-efficient firms expect to earn the highest profits in the market.

The main reason why the monopoly result does not generalize is that in an auction with multiple licenses, a strategic effect appears which may change the balance of forces at the auction stage. The monopoly result rests on the fact that a more efficient monopolist can make more profit in the aftermarket than any other (less efficient) firm can make. This effect is also present in an oligopoly context if we fix the aftermarket strategies of other winning firms. We call this the direct effect: a more cost efficient firm will make *ceteris paribus* more profit in the aftermarket, and is therefore willing to bid more in the auction than a less efficient firm is willing to bid. The strategic effect, which arises in oligopoly markets, goes against this direct effect.

The strategic effect is present in almost all market settings and originates from two sources: (*i*) a negative externality stemming from the fact that any firm prefers to compete with firms that are less cost-efficient, and (*ii*) *ex-post* correlation of firms' types stemming from the fact that all winning firms have submitted bids that are higher than bids of any loosing firms. Depending on the market conditions and the *ex-ante* distribution of firms' costs, the strategic effect can be so strong that the benefit from being more efficient, *i.e.*, the direct effect, is outweighed by the losses due to tougher competition, *i.e.*, the strategic effect. When this occurs, the most cost-efficient firms make less profit in the aftermarket when they compete with each other than the least cost-efficient firms do. In this case, auction efficiency, *i.e.*, allocating licenses to the most profitable firms, and social efficiency, *i.e.*, allocating licenses to the most cost-efficient firms, work in opposite directions. We study market conditions under which auctions (may) select inefficient firms.

More technically, we consider a standard multi-unit uniform-price auction where firms have private information about their costs, and overall economic efficiency requires the most efficient firms to win the auction. A strategy for the firms is a function specifying how a firm's bid depends on its efficiency parameter. The generalization of the monopoly result to the case where multiple licenses are auctioned requires that the more efficient a firm, the higher it bids in the auction, *i.e.*, there should exist a monotone symmetric bidding equilibrium where a firm's bidding strategy is increasing in its efficiency parameter. We call such an equilibrium an "efficient selection equilibrium". Such an equilibrium can only exist when a firm's expected aftermarket profit conditional on winning the auction also increases in its efficiency parameter. We identify market conditions under which firms that are more cost-efficient do not make the highest profits, implying that an efficient selection equilibrium does not exit.

A first, more easily identifiable condition under which an efficient selection equilibrium fails to exist is that firms' efficiency parameters are positively correlated (affiliated) so that learning one's own efficiency parameter provides information about other firms' private information. In practice, positive affiliation of firms' efficiency parameters may naturally arise in sectors where firms use similar production technologies and prices of inputs fluctuate with (macroeconomic) shocks that are common to all firms. Alternatively, firms may implement cost-saving technologies that arise from an exogenous stochastic process. In both cases, if a firm is more cost-efficient itself, it infers that all other firms are more likely to be cost-efficient as well. Therefore, firms that are more cost-efficient expect to be competing with other more cost-efficient firms which are known to be fierce competitors. We show that for *any* oligopolistic market, no matter how weak the negative externality is, there are distributions of firms' types for which an efficient selection equilibrium does not exist.

A second, more surprising condition under which an efficient selection equilibrium fails to exist is where firms' efficiency parameters are *ex-ante* independent. Despite firms' types being *ex-ante* independent, the types of firms that win licenses are correlated. This is so because all winning firms outbid the firm with the highest losing bid.[4] As this *ex-post* form of positive correlation (affiliation) actually is the only correlation that is relevant for determining the optimal bidding strategy, the intuitive reason for the nonexistence of an efficient selection equilibrium is then the same as above. We show that in this case of statistically independent types, an increasing equilibrium fails to exist only if the negative externality is sufficiently strong.

When one of these two conditions holds, an efficient selection equilibrium does *not* exist. This implies that only (*i*) asymmetric equilibria exist in which different firms have different bidding functions, or (*ii*) the equilibrium bidding functions are not monotone, or (*iii*) firms use random bidding strategies, or (*iv*) a decreasing equilibrium exists. In all of these four cases, there is at least a positive probability that less efficient firms will bid more than more efficient firms and therefore obtain the licenses. Thus, if firms' types are highly correlated, or the negative externality is strong enough, the overall welfare outcome is inefficient with positive probability. In other words, in an increasing bidding equilibrium there is a kind of winner's curse or adverse selection effect present, the strength of which depends on the type of bidder. Bidders optimally "adjust" their bids for this effect and as the effect is stronger for firms that are more efficient, highly efficient firms may "adjust" their bid so much that an efficient selection equilibrium fails to exist.

Our last result, which we show by means of an example, is that when firms' types are correlated *and* the negative externality is strong enough, there exists a unique monotone symmetric bidding equilibrium that is *decreasing*. In this case, firms that are most profitable in the aftermarket and that submit the highest bids for licenses are the least cost-efficient firms. Thus, despite the auction itself being efficient, the allocation is the most inefficient, implying lower social welfare than from *any other* selection mechanism.

The rest of the paper is organized as follows. As there is an extensive relatively recent literature that is related to this paper, we start Section 2 with a review of existing literature. Section 3 describes the two-stage model with an auction stage and a market competition stage. Section 4 briefly derives the benchmark monopoly result when only one license is auctioned, and Section 5 provides necessary and sufficient conditions for

[4] An easy way to understand the way this ex-post correlation arises is to consider a case where 3 firms compete for 2 licenses and consider the possibility that an efficient selection equilibrium exists. If a firm with a certain efficiency level wins the auction, the firm then knows that it cannot be the case that both other competing firms are more efficient than this firm. This information creates *ex-post* correlation, which firms must take into account in the auction stage when calculating their bids.

an efficient selection equilibrium to exist with multiple licenses. We also illustrate what these general conditions imply in case of commonly used Bertrand and Cournot models. Section 6 presents an example where the unique monotone symmetric bidding equilibrium is decreasing. Section 7 concludes and provides a discussion of remaining issues. The appendix contains all proofs.

2. Literature review

There is a relatively large, recent literature on the possibility of inefficient allocation of licenses in auctions due to the presence of externalities. First, there is a literature where one license is auctioned and the auction winner competes in the aftermarket with non-winners.[5] Moldovanu and Sela (2003) analyze aftermarket Bertrand competition where cost is private information at the auction stage.[6] When in such a situation a patent for a cost-reducing technology is auctioned amongst the competitors, they show that standard auction formats do not exhibit efficient equilibria where bids are increasing in the firms' efficiency parameter. Goeree (2003) and Das Varma (2003) analyze a similar setting but allow for signaling private information through the auction bid. Das Varma (2003) shows that when aftermarket competition is in strategic substitutes (Cournot competition) an efficient equilibrium does exist as firms have an incentive to overstate their efficiency through higher bids. When aftermarket competition is in strategic complements (Bertrand competition) this is not the case, however, as firms have an incentive to understate their efficiency and will adjust their bids downwards. Therefore, under Bertrand competition, an efficient equilibrium may fail to exist. Goeree (2003) mainly focuses on the comparison of firms' bid with and without signaling under different auction formats, and on the seller's revenue in case a separating (increasing) equilibrium exists. Also in his model, however, there is a possibility for efficient equilibria not to exist, namely if players have an incentive to understate their valuation. Katzman and Rhodes-Kropf (2008) show how different bid-announcement policies affect an auction's revenue and efficiency.

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[5] Jehiel and Moldovanu (2006) provide an overview of existing work in this area and argue that in case firms' aftermarket profits depend on private information in the hands of other winning firms there is an informational externality. See, also, Jehiel *et.al.* (1996) and Jehiel and Moldovanu (2000) for related papers where an (informational) externality may lead to inefficiency even in standard single-unit auctions.

[6] To avoid signaling issues, they consider the case where the true production costs of the bidding firms are revealed after the auction. This creates the required for the results negative externality.

The above papers show that *auction* inefficiencies may already occur in single-object auctions with interdependent valuations. There are some obvious differences between our paper and the papers mentioned above as we consider auctions where multiple licenses are offered for sale and as we abstract from signaling issues. The reason why an efficient equilibrium does not exist in Moldovanu and Sela (2003) is that on the margin, due to the strong negative externality, a firm that is more efficient has a lower willingness to pay for a license than a firm that is less efficient. We use a similar reasoning to prove Proposition 2 where we show that when types are *ex ante* independently distributed, an increasing equilibrium may fail to exist. In our context, we also need that the strategic effect is sufficiently strong for this result. The results on signaling in Goeree (2003) and Das Varma (2003) suggest that an increasing equilibrium may fail to exist also in our framework if the bids are made public after the auction and bidders compete in prices.

Compared to these papers, we offer two other reasons why *market* inefficiencies may arise. First, an efficient equilibrium may fail to exist if types are *ex ante* correlated. This failure may even occur when the strategic interaction between firms in the aftermarket is limited. *Ex-ante* correlation of types guarantees that the strategic effect is strong. In this case, the *market* inefficiency is due to the inefficiency of the auction. Second, and more importantly, when both *ex-ante* correlation of types and the market externality are strong enough, expected profit of the winning firms decreases in their efficiency parameter, and the only monotone equilibrium of the auction is a decreasing equilibrium. This equilibrium guarantees an efficient allocation of licenses in the auctions as the firms that value the licenses most also win the auction. However, the efficient *auction* allocation leads to an inefficient *market* allocation as the most inefficient firms win the licenses. Therefore, our paper points to the fact that inefficient market allocations may not only be due to inefficiencies created in the process allocating licenses.

There is also a literature on inefficiencies created by multiple license auctions if players have interdependent valuations,[7] but these papers usually address issues related to collusion and/or asymmetries between different participants. Hoppe *et al.* (2006), for example, consider an auction with asymmetric players (incumbents and entrants) and focus on the competitiveness induced by the number of licenses that is auctioned.

[7] Krishna (2002), Example 6.4, shows that it might happen that the bidder with the highest type is the bidder with the lowest valuation and that standard auction formats do not have efficient equilibria.

Contrary to common wisdom, they show that auctioning more licenses does not need to result in a more competitive final outcome due to the asymmetries between players. Grim *et al.* (2003) study conditions under which firms may collude on a low price equilibrium. Our paper, in contrast, focuses on inefficiencies that may arise when selling multiple licenses when players are *ex-ante* identical and collusion is not an issue.

3. The Model

Access to the aftermarket is limited to the firms that have obtained licenses to operate in the market. The government allocates $n \geq 2$ licenses in a multi-unit auction to the highest bidding firms, and we assume that $N \geq (n+1)$ firms participate in the auction. The assumption that firms can obtain at most one license is motivated by the fact that this rule has been used in many recent UMTS auctions as a way to guarantee aftermarket competition. In the oligopolistic aftermarket, firms compete by simultaneously choosing a value of the strategic variable *s*. Depending on the market, we can interpret *s* as either a price *p* or a quantity *q,* or any other relevant strategic variable. The profit π^i of firm *i* is determined by the level of *s* that firm *i* and the other (*n* - 1) firms choose, and by the firm's efficiency parameter e_i. We assume that π^i is symmetric in all s_j, $j \neq i$, and we write it as

$$\pi^i = \pi(s_i, s_{-i}, e_i),$$

where s_{-i} denotes levels of strategic variables chosen by all other firms. To shorten notation, we denote the partial derivatives of the function $\pi(s_i, s_{-i}, e_i)$ and other functions by subscripts as follows:

$$\pi_i \equiv \partial\pi / \partial s_i,\ \pi_j \equiv \partial\pi / \partial s_j \text{ for } j \neq i,\ \pi_e \equiv \partial\pi / \partial e_i,\ \pi_{i,j} \equiv \partial^2\pi / \partial s_i \partial s_j, \textit{ etc.}$$

The efficiency parameter e_i positively influences the profit of firm *i* reducing its fixed as well as marginal costs. Therefore, for a typical cost function $c(e,q)$ of a firm we have:

$$c_e(e,0) \leq 0,\ c_{e,q}(e,q) \leq 0.$$

Under a natural assumption of positive marginal cost, *i.e.*, $c_q(e,q) > 0$, it follows that the efficiency parameter unambiguously reduces the total (fixed plus variable) costs, *i.e.*, that $c_e(e,q)<0$ for $q>0$. The efficiency parameter can also be more broadly interpreted as a parameter, *e.g.*, advertisement efficiency, which enhances firms' profit possibilities, *i.e.*, $\pi_e>0$.

When firms compete in quantities (Cournot competition), $s_i=q_i$ and firms' profit function is given by

$$\pi(q_i,q_{-i},e_i)=q_i p\left(\sum_j q_j\right)-c(e_i,q_i),$$

so that $\pi_e=-c_e>0$ and $\pi_{i,e}=-c_{e,q}>0$. When firms compete in prices (differentiated Bertrand competition), $s_i=p_i$ and firms' profit function is given by

$$\pi(p_i,p_{-i},e_i)=q(p_i,p_{-i})p_i-c(e_i,q(p_i,p_{-i})),$$

where $q(p_i,p_{-i})$ represents firm *i* 's demand, so that $\pi_e=-c_e>0$ and $\pi_{i,e}=-c_{e,q}q_i\geq 0$.

In order to ensure the existence, uniqueness and stability of the (Bayes-) Nash equilibrium in the aftermarket, firms' marginal profit function must satisfy a stability requirement. We follow Bulow *et al.* (1985) and assume in the case of strategic complements, where $\pi_{i,j}>0$, that $\left(\pi_{i,i}+\sum_{k\neq i}\pi_{i,k}\right)<0$ and in case of strategic substitutes, where $\pi_{i,j}<0$, that $\left(\pi_{i,i}-\pi_{i,j}\right)<0$.

We analyze the case where the government organizes a multi-unit uniform-price auction to allocate the *n* licenses, where all the winning firms pay the same license fee *w,* which is equal to the highest non-winning bid. This uniform-price auction allows us to simplify the exposition of results while keeping the formulation of the aftermarket competition stage quite general.[8] In the main body of the paper we assume that resale of licenses is not allowed. In the final section, we discuss how allowing for different auction formats and resale of licenses after the auction will affect our results.

A firm's efficiency parameter e_i, *i.e.*, the type of firm *i*, is its private information in the auction stage. The *prior* joint distribution of types is symmetric and denoted by *F*. This distribution has a finite support $[\underline{e},\overline{e}]$, and is assumed to be weakly affiliated (thus allowing for statistical independence). A firm *i* submits a bid b_i based on e_i. We denote a monotone symmetric equilibrium bidding function by $b(e)$, so that firm *i* bids $b(e_i)$ in equilibrium.

[8] Other auctions' formats and other informational scenarios (see further) in our model can only be effectively analyzed when the efficiency parameter has a vanishingly small variation.

Depending on what information is revealed immediately after the auction is held, three different scenarios can be considered:

a) A *private information* scenario, where neither firms' types nor the winning bids become public.

b) An *imperfect information* scenario, where only the bids of the winning firms but not their types become public.

c) A *full information* scenario, where types of all winning firms become public.

In what follows, we mainly focus on the private information scenario. One reason for doing so is to bring out most sharply the result that inefficiencies are due to strategic interaction and not to signaling. Signaling plays an important role in the imperfect information scenario and the auction stage then has to be analyzed as an N-player signaling game with each firm being a sender and a potential receiver of signals. The results of Das Varma (2003) suggest that the signaling effect helps the efficient equilibrium to survive in case of strategic substitutes, but destroys existence in case of strategic complements. The full information scenario, on the other hand, does not seem to be realistic. Moreover, this scenario can be analyzed in a way similar to the private information scenario.

In the private information scenario, a type e_i paying a license fee w and choosing s_i in the aftermarket has an expected market profit conditional on winning the auction of

$$E\left(\pi\left(s_i, s_{-i}, e_i\right) \middle| e_i, b\left(e_l\right) \leq w < b\left(e_k\right)\right),$$

where we indicate all winning firms other than firm i by the index k and all losing firms by index l. Maximizing expected profit with respect to s_i and assuming that all other winning firms choose s_k optimally yields the first-order conditions defining firms' aftermarket Nash equilibrium strategy. In case the bidding function $b(e)$ monotonically increases, we denote it as $b^{(+)}(e)$, and write these first-order conditions as follows:

$$0 = E\left(\pi_i\left(s_i^*, s_{-i}^*, e_i\right) \middle| e_i, e_l \leq z < e_k\right),\ i = 1, \ldots, n\ ,$$

where z denotes the type of firm that submits the n^{th} highest bid among $(N - 1)$ firms other than firm i determining the license fee w, *i.e.*, $w = b^{(+)}(z)$. Under the assumptions that we made about the profit function $\pi(s_i, s_{-i}, e_i)$, these first-order conditions uniquely define a market stage Nash equilibrium strategy $s_i^* = s(e_i, z)$.

4. The Monopoly benchmark

If only one license is auctioned, *i.e.*, $n = 1$, the auction stage of the game can be characterized by a second-price private-valuation auction. In such an auction, it follows from standard results that each firm bids its monopoly profit. Denoting the market-stage profit of firm i by $\pi^i = \pi(s_i, e_i)$, the first-order condition that determines the monopoly level of strategic variable $s^*(e_i)$ is

$$0 = \pi_i(s^*(e_i), e_i).$$

The monopoly profit of firm i with efficiency e that chooses an optimal level of its strategic variable is given by $\pi^M(e) = \pi(s^*(e), e)$. Differentiating it with respect to the efficiency parameter e and using the above first-order condition (or, the envelope theorem) yields the monopoly result:

$$d\pi^M / de = \pi_e > 0.$$

This implies that the most efficient firm expects to earn the highest monopoly profit in the aftermarket and, consequently, submits the highest bid. This is the monopoly result alluded to in the Introduction.

5. When an increasing bidding equilibrium does not exist

We are now ready to discuss the oligopoly case and analyze the necessary and sufficient conditions for an efficient selection equilibrium to exist. We first derive these conditions for the general case described in Section 3. Then we analyze two sets of circumstances (firms' types being independently distributed, and affiliated types) under which the necessary conditions cannot be satisfied so that the auction stage does not have an efficient selection equilibrium. With independent types, we also indicate what these conditions imply in case of Bertrand and Cournot competition.

Let $b^{(+)}(e)$ be a strictly increasing symmetric equilibrium bidding function and $s_i^* = s(e_i, z)$ be the corresponding firms' aftermarket Nash equilibrium strategy. Denoting a firm's reduced-form profit by $\overline{\pi}(e_i, e_{-i}, z)$, *i.e.*,

$$\overline{\pi}(e_i, e_{-i}, z) \equiv \pi(s_i^*, s_{-i}^*, e_i),$$

allows us to write the expected profit of type x conditional on getting a license as

$$v^{(+)}(x, z) \equiv E\left(\overline{\pi}(e_i, e_{-i}, z) \middle| e_i = x, e_l \leq z < e_k\right).$$

The function $v^{(+)}(x, z)$ is a firm's valuation function, which is used in the auction stage to determine the optimal bidding strategy. The following proposition derives an equilibrium bidding function and necessary and sufficient conditions for an increasing symmetric bidding equilibrium to exist.

Proposition 1. *If $v_x^{(+)} > 0$ and $v_x^{(+)}(e,e) + v_z^{(+)}(e,e) > 0$ for all $e \in [\underline{e}, \overline{e}]$, then there exists a unique symmetric strictly increasing bidding equilibrium given by $b^{(+)}(e) = v^{(+)}(e,e)$. If such an equilibrium exists, then $v_x^{(+)}(e,e) \geq 0$ and $v_x^{(+)}(e,e) + v_z^{(+)}(e,e) \geq 0$ for all $e \in [\underline{e}, \overline{e}]$.*

These statements can be explained as follows. Suppose that $x = z$. In other words, suppose that firm i and another firm, let us say firm m, have the same type, *i.e.*, $e_i = x = z = e_m$, so that they together determine the auction price w, and $e_k > z$ for all other winning firms k and $e_l < z$ for all other losing firms l. In this case, firms i and m compete for only one license. In equilibrium, they bid their entire expected market profits $v^{(+)}(z, z)$ and the equilibrium bidding function must satisfy $b^{(+)}(z) = v^{(+)}(z, z)$.

Suppose now that x is marginally larger than z. Then, in order to get a license, firm i must get a marginally higher expected profit $v^{(+)}(x, z)$ than firm m, so that firm i *can* bid marginally higher than firm m. Thus, $v^{(+)}(x, z)$ must be a weakly increasing function of x at $x = z$, *i.e.*, $v_x^{(+)}(e,e) \geq 0$ is the first necessary condition. The other necessary condition $v_x^{(+)}(e,e) + v_z^{(+)}(e,e) \geq 0$ guarantees that the actual bid $v^{(+)}(x, x)$ of firm i is *indeed* not lower than the bid $v^{(+)}(z, z)$ of firm m.

On the other hand, if $b^{(+)}(e) = v^{(+)}(e,e)$ is a strictly increasing bidding function, the sufficient condition $v_x^{(+)} > 0$, which basically is the second-order condition for profit maximization, guarantees that a firm has no profitable deviation from $b^{(+)}(e)$. In accordance with Proposition 1, the standard assumptions of the affiliated valuation model (Milgrom and Weber, 1982), *i.e.*, $v_x^{(+)} > 0$ and $v_z^{(+)}(e,e) \geq 0$, always lead to the existence and uniqueness of an efficient equilibrium.

From the point of view of a winning firm *i*, the types of all other (*n* - 1) winning firms are affiliated even if the types are *ex-ante* independent. The degree of this affiliation, in general, is determined by firm *i*'s own type and by the *ex-ante* distribution of types. In the limit case, when types of all competitors of firm *i* are perfectly correlated, *i.e.*, when $e_k = z$ for all *k*, the partial derivatives of the reduced-form profit function $\overline{\pi}$ can be analytically calculated. This is the content of the following lemma.

Lemma 1. *In case $e_k = z$ for all k, the partial derivatives of the reduced-form profit function of firm i $\overline{\pi}(e_i, e_{-i}, z)$ at $e_i = e_j = z = x$ are:*

$$\overline{\pi}_i(x,x,x) = \pi_e > 0,$$

$$\overline{\pi}_j(x,x,x) = -\frac{\pi_j \pi_{i,e}}{\pi_{i,i}} < 0 \text{ and}$$

$$\overline{\pi}_z(x,x,x) = \frac{(n-1)^2 \pi_j \pi_{i,j} \pi_{i,e}}{(\pi_{i,i} + (n-1)\pi_{i,j})\pi_{i,i}}.$$

Lemma 1 turns out to be useful in the rest of this section. Using a continuity argument, we argue that the inequalities $\overline{\pi}_i > 0$ and $\overline{\pi}_j < 0$ also hold when competitors' types are highly (but not perfectly) correlated, and when *x* and *z* are close to each other (but do not coincide). Hence, if the *ex-post* correlation of all winning firms' types is large, the direct effect $\overline{\pi}_i$ is always positive, *i.e.*, each firm wants to be more efficient, and the indirect effect $\overline{\pi}_j$ is negative, *i.e.*, each firm wants to compete with less efficient firms.

When firms compete in prices (the case of strategic complementarity), Lemma 1 can be generalized: the indirect effect $\overline{\pi}_j$ is always negative, not only for $e_i = e_j = z = x$. If, for example, firm *i* becomes more efficient, the expected efficiency of any of its competitors also increases and they are expected to charge lower prices. Consequently, firm *i* lowers its price, which, due to strategic complementarity, forces its competitors to lower their prices even further. This is not the case when firms compete in quantities (the case of strategic substitutes). When firm i becomes more efficient, so do (in expected terms) all its competitors, and *ceteris paribus* they expand their outputs (due to the direct effect). This reduces the profit of firm i. However, every competitor of firm i also faces more intense competition which makes them to reduce their own outputs. Due to this second channel, a higher efficiency of competitors of firm i has a positive effect on the profit of firm i. In general, the indirect effect can be either positive or negative, depending on which of these effects is stronger for the realized values of firms' types. Nevertheless, Lemma 1 states that also in case of strategic substitutes the indirect effect is negative when firms' types are close to each other.

This indirect effect, or negative externality, is the origin of a "strategic effect" in the auction stage of the game. The more efficient a firm, the more efficient its competitors are expected to be in the aftermarket, due to the *ex-post* correlation. This has a negative effect on the firm's market profit, due to the negative externality. The combination of *ex-post* correlation and the negative externality together is what we call the "strategic effect". Thus, the two effects determining bidding behavior in the auction stage, work in opposite directions: the positive direct effect forces a firm to bid *higher* if it is more efficient, whereas the negative strategic effect forces a firm to bid *lower* if the firm is more efficient.

An interesting consequence of Lemma 1 is that although the auction price $w = b^{(+)}(z)$ is a sunk cost in the aftermarket stage, it affects the distribution of a firms' competitors and, consequently, influences a firm's profit indirectly through their aftermarket strategy due to $\overline{\pi_z} \neq 0$.[9]

Statistically independent type

Using Proposition 1 and Lemma 1 as a general tool, we now first analyze the case of independent types. Let firms' efficiency parameter e_i be identically and independently distributed over $[\underline{e}, \overline{e}]$ in accordance with an arbitrary differentiable distribution function $F(e)$ with corresponding density $f(e) \equiv F'(e)$ and hazard rate $h(e) \equiv f(e)/(1-F(e))$. The following proposition states when an efficient selection equilibrium fails to exist.

[9] In alternative settings, Janssen (2006) and Janssen and Karamychev (2007) also show that the sunk auction price may affect firms' aftermarket behavior.

Proposition 2. *Let* γ *be defined as* $\gamma \equiv \lim_{e \to \overline{e}} (\overline{e} - e) \cdot h(e) \geq 1$.[10] *If the condition*

$$\frac{\gamma}{\gamma+1} \frac{(n-1)\pi_j \pi_{i,e}}{(\pi_{i,i} + (n-1)\pi_{i,j})\pi_e} > 1$$

holds at $e_i = e_j = \overline{e}$*, then an efficient selection equilibrium does not exit.*

As $\gamma/(\gamma+1) < 1$,[11] the condition of Proposition 2 can only be satisfied when

$$\pi_e - \frac{(n-1)\pi_j \pi_{i,e}}{(\pi_{i,i} + (n-1)\pi_{i,j})} < 0 .$$

One can verify that the latter condition holds if and only if an industry-wide increase in the common efficiency of all competitors in an oligopoly market results in a fall in the industry-wide profits.

In order to better understand when this condition is satisfied, let the efficiency parameter only affect firms' costs but not their revenues, so that

$$\pi(s_i, s_{-i}, e_i) = R(s_i, s_{-i}) - c(e_i, q_i),$$

and $R(s_i, s_{-i})$ is revenue. In this case, the condition of Proposition 2 can be interpreted as a restriction on the cost function:

$$\frac{c_e}{c_{e,q}} < \frac{\gamma}{\gamma+1} \frac{(n-1)\pi_j}{(\pi_{i,i} + (n-1)\pi_{i,j})} \frac{\partial q}{\partial s_i} .$$

In words, the larger the impact of the efficiency parameter on marginal cost relative to its impact on total costs, the more likely it is that an efficient selection equilibrium does not exist.

We illustrate Proposition 2 by considering some specific functional forms under differentiated Bertrand and Cournot competition. The examples focus on specifications where demand is linear and downward sloping (to get the usual upward (downward) sloping reaction functions under Bertrand (Cournot) competition. In this case, we show that some convexity in the cost function, *i.e.*, decreasing returns to scale, may already be enough to get the non-existence of an efficient selection equilibrium.

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[10] The fact that $\gamma \geq 1$ is easily seen by applying l'Hopitals' rule. If $f(\overline{e}) > 0$, then $\gamma = 1$. If $f(\overline{e}) = 0$, then it must be that $f'(\overline{e}) < 0$ so that $\gamma > 1$.

[11] The term $\gamma/(\gamma+1)$ represents the strength of the *ex-post* affiliation. Proposition 2 uses the fact that if z is close to the upper end of the distribution $\overline{e}$, this affiliation is reasonably strong because the types of all winning firms will be between z and $\overline{e}$. The strongest *ex-post* correlation is achieved in the limit when γ unboundedly increases so that $\gamma/(\gamma+1)$ asymptotically approaches 1.

Example 1: Differentiated Bertrand competition.

Let the symmetric cost function of firms be given by $c(e,q)=q^{\alpha}(1-e)$, *with* $\alpha\geq 1$. *When* $\alpha=1$, *marginal cost is constant whereas* $\alpha>1$ *represents convex costs. For this cost function, the ratio* $c_e/c_{e,q}$ *mentioned above is given by* q/α *so that Proposition 2 is easier to satisfy for larger values of* α.

Let the efficiency parameters of firms be identically and independently distributed over the $[0,1]$ *interval in accordance with the distribution function* $F(e)=1-(1-e)^{\gamma}$,[12] *with* $\gamma\geq 1$. *When* $\gamma=1$ *the distribution is uniform whereas* $\gamma>1$ *represents decreasing density distributions.*

Finally, let firms' demand functions q_i *be linear and given by*

$$q_i(p_i,p_{-i})=1-p_i+b\sum\nolimits_{j\neq i}p_j,$$

where $b\in[0,1/(n-1)]$. *The range of values* b *can take on guarantees that aggregate demand* $Q(p_1,\ldots,p_n)\equiv\sum_i q_i(p_i,p_{-i})$ *is not increasing in prices. The case* $b=1/(n-1)$ *corresponds to the unit demand, i.e., consumers always buy the product from one of the firms, and the prices only determine firms' market shares. When* $b<1/(n-1)$, *aggregate demand strictly decreases in prices.*

Firm's profit function is given by

$$\pi(p_i,p_{-i},e_i)=p_i\left(1-p_i+b\sum\nolimits_{j\neq i}p_j\right)-\left(1-p_i+b\sum\nolimits_{j\neq i}p_j\right)^{\alpha}(1-e_i),$$

so that the best response functions $p_i(p_{-i},e_i)$ *are upward sloping and implicitly defined by*

$$0=1-2p_i(p_{-i},e_i)+b\sum\nolimits_{j\neq i}p_j+\alpha\left(1-p_i(p_{-i},e_i)+b\sum\nolimits_{j\neq i}p_j\right)^{\alpha-1}(1-e_i).$$ [13]

It is then easy to see that that Nash equilibrium prices at $e_i=e_j=1$ *are*

$$p_i=p^*\equiv 1/(2-(n-1)b).$$

Inspection of the partial derivatives of the profit function π *at this point yields:*

$$\pi_{i,i}=-2,\ \pi_{i,j}=b,\ \pi_j=b/(2-(n-1)b),\ \pi_e=(2-(n-1)b)^{-\alpha},\ \text{and}$$

$$\pi_{i,e}=-\alpha(2-(n-1)b)^{1-\alpha}.$$

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[12] It can be verified that indeed $\lim_{e\to 1}(1-e)\cdot h(e)=\gamma$ for this distribution, so that the notation is consistent.

[13] The positive slope of the reaction function follows from the fact that for $\alpha\geq 1$, the right-hand side of this equation monotonically decreases in p_i and monotonically increases in p_j.

Substituting these values into the condition of Proposition 2 yields that if

$$b > \frac{2(\gamma+1)}{(n-1)(\gamma\alpha+\gamma+1)},$$

an efficient selection equilibrium does not exist. For example, in case of unit demand ($b = 1/(n-1)$) and uniform distribution ($\gamma = 1$), an increasing equilibrium fails to exist for $\alpha > 2$. The condition can also be satisfied for any values of $b > 0$, γ and n provided firms' cost is sufficiently convex, i.e., when $\alpha > (2 - b(n-1))(\gamma+1)/(\gamma b(n-1))$. For completely independent markets ($b = 0$), however, the condition always fails as this coincides with the monopoly result.

Example 2: Cournot competition.

We use the same cost and distribution function as in the previous example to show that the non-existence result illustrated there also holds under quantity competition. Let market demand be linear and given by $Q = 1 - p$. Under Cournot competition, a firm's profit function is given by

$$\pi(q_i, q_{-i}, e_i) = q_i\left(1 - \sum\nolimits_j q_j\right) - q_i^{\alpha}(1 - e_i).$$

Best response functions $q_i(q_{-i}, e_i)$ are downward sloping and implicitly defined by

$$0 = 1 - 2q_i(q_{-i}, e_i) - \sum\nolimits_{j \neq i} q_j - \alpha\left(q_i(q_{-i}, e_i)\right)^{\alpha-1}(1 - e_i).$$ [14]

It is then easy to see that equilibrium output levels at $e_i = e_j = 1$ are given by

$$q_i = q_j = 1/(n+1).$$

Inspection of the partial derivatives of the profit function at this point yields:

$$\pi_{i,i} = -2,\ \pi_{i,j} = -1,\ \pi_j = -1/(n+1),\ \pi_e = 1/(n+1)^{\alpha},\ \text{and}\ \pi_{i,e} = \alpha/(n+1)^{\alpha-1}.$$

Substituting these values into the condition of Proposition 2 yields that if

$$\alpha > \frac{(n+1)(\gamma+1)}{(n-1)\gamma},$$

an efficient selection equilibrium does not exist. It is easy to see that if α is sufficiently large so that the cost function is sufficiently convex, this condition is satisfied for all feasible values of the other parameters. For example, it is easily verified that if $\alpha > 6$, the condition is satisfied for all feasible values of the other

[14] The negative slope of the reaction function follows from the fact that for $\alpha \geq 1$, the right-hand side of this equation monotonically decreases in both q_i and q_j.

parameters, i.e., for $n \geq 2$ *and* $\gamma \geq 1$*. When α is smaller, the conditions on n and/or γ become more tight. For example, in case of quadratic cost (*$\alpha = 2$*) and a linearly decreasing density function* $f(e) = 2(1-e)$ *(*$\gamma = 2$*), an increasing equilibrium fails to exist for* $n \geq 8$ *licenses, whereas in case of cubic cost (*$\alpha = 3$*) this happens already for* $n \geq 4$*. It is also worth to note that in case of linear demand and constant marginal cost (*$\alpha = 1$*), we cannot show that an efficient selection equilibrium fails to exist because the above condition is never satisfied.*

In both examples, individual and market demand functions have been assumed to be linear. The examples indicate that in this case, cost needs to be convex (decreasing returns to scale). Similar (although slightly more extensive) derivations show that for convex demand functions the corresponding sufficient conditions can be relaxed so that they also cover increasing returns to scale technologies. The examples also indicate that an efficient selection equilibrium is more likely not to exist, if the number of licenses to be allocated is larger.

Affiliated types

In the examples above where firms' types are assumed to be statistically independent, the negative externality must be sufficiently strong in order for an efficient selection equilibrium not to exist. It turns out that that if firms' types are *ex-ante* correlated, an increasing equilibrium may fail to exist even if the externality is weak.[15] We illustrate the case of *ex-ante* affiliation by means of the following example.

Example 3: Affiliated types and an arbitrarily weak externality.

We assume conditions that commonly hold in oligopoly markets, namely $\overline{\pi_i} > 0$*, i.e., firms prefer being more cost-efficient themselves, and* $\overline{\pi_j} < 0$*, i.e., firms prefer competing with less cost-efficient competitors. Then, suppose that the economy can be in either one of the following two states with equal probability: a high-variation state in which firms' efficiency parameters are independently and uniformly distributed over the* $[0,2]$ *interval with mean* 1 *and variance* $1/3$*, and a*

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[15] It is easy to see that if the externality is completely absent, *i.e.*, if $\overline{\pi_j} = \overline{\pi_z} = 0$, firms have local monopolies and the monopoly result continues to hold.

low-variation state in which they are independently and uniformly distributed over an interval $[\delta - \varepsilon, \delta + \varepsilon]$ *with mean* $\delta < 1$ *and variance* $\varepsilon^2 / 3$. *For simplicity, we consider the case of two licenses, and assume that* δ *is small.*

If the realized type of firm i is $e_i^1 = \delta$, *the economy is in either state with equal probability, and the expected type of its competitor is*

$$E\left(e_j \middle| e_i = e_i^1\right) = (1+\delta)/2 .$$

If, to the contrary, the type of firm i is $e_i^2 = \delta + 2\varepsilon > e_i^1$, *the economy is in the high-variation state with probability one, and the expected type of its competitor is*

$$E\left(e_j \middle| e_i = e_i^2\right) = 1 > (1+\delta)/2 = E\left(e_j \middle| e_i = e_i^1\right).$$

In other words, the more efficient type e_i^2 *faces in expected terms more intense competition but is also more efficient itself than the low-efficient type* e_i^1.

It is easy to see that when the variation parameter ε *reduces to zero so that* e_i^2 *approaches* e_i^1 *from above, the direct effect for both types becomes equal because both types* e_i^2 *and* e_i^1 *become equally efficient in the limit. The strength of the strategic effect, however, remains very different. Indeed, with probability ½ both types are in the same high-variation state and with probability ½ they are in different states. In the latter case, they expect to compete with rivals of different types: type* e_i^1 *expects to compete with type* $e_j = \delta$ *whereas type* e_i^2 *expects to compete with type* $e_j = 1$. *Due to the negative externality (*$\overline{\pi_j} < 0$*), type* e_i^1 *gets in the limit strictly higher expected profit than type* e_i^2, *even though* $e_i^2 > e_i^1$ *(for sufficiently small* δ*). This implies that the valuation and, hence, the bid of type* e_i^2 *is lower than of type* e_i^1, *and the bidding function cannot be monotonically increasing.*

In this example, the distribution of firms' types is such that the expected efficiency level of a rival firm conditional on its own efficiency level is very sensitive to the own efficiency level, *i.e.*, the derivative $dE\left(e_j \middle| e_i\right)/ de_i$ is large. Typically, any distribution with positive correlation (affiliation) and a density function $f\left(e_j \middle| e_i\right)$ that is discontinuous (or "almost" discontinuous) in e_i will be suitable to make this point.

6. On decreasing bidding equilibria

The analysis in the previous section leads us to a natural question, namely whether there exist market structures and distributions of types for which not only an increasing equilibrium fails to exist, but instead a symmetric decreasing equilibrium does exist. In such an equilibrium, the least-efficient firms always submit the highest bids and obtain the available licenses. In this section, we first argue that both a negative externality and *ex-ante* affiliation of firms' types are necessary for a decreasing equilibrium to exist. Next, we provide an example of specific market conditions under which the strategic effect is strong enough so that a decreasing equilibrium indeed does exist.

When a monotone symmetric equilibrium bidding function is a decreasing function, firm's valuation function $v^{(-)}(x,z)$ must be defined as follows

$$v^{(-)}(x,z) \equiv E\left(\overline{\pi}(e_i, e_{-i}, z) \middle| e_i = x, e_k < z \le e_l\right).$$

The following proposition derives an equilibrium bidding function and necessary and sufficient conditions for a symmetric decreasing bidding equilibrium to exist.

Proposition 3. *If* $v_x^{(-)} < 0$ *and* $v_x^{(-)}(e,e) + v_z^{(-)}(e,e) < 0$ *for all* $e \in [\underline{e}, \overline{e}]$*, then there exists a unique symmetric strictly decreasing bidding equilibrium given by* $b^{(-)}(e) = v^{(-)}(e,e)$*. If a decreasing equilibrium exists, then* $v_x^{(-)}(e,e) \le 0$ *and* $v_x^{(-)}(e,e) + v_z^{(-)}(e,e) \le 0$ *for all* $e \in [\underline{e}, \overline{e}]$.

The proof of Proposition 3 is similar to the proof of Proposition 1 and is, therefore, omitted. Consider first the condition $v_x^{(-)} < 0$. If firms types are independent, then

$$v_x^{(-)}(x,z) \equiv E\left(\overline{\pi_i}(x, e_{-i}, z) \middle| e_k < z \le e_l\right).$$

Under the assumption that firms prefer being more cost-efficient themselves we have that $\overline{\pi_i}(x, e_{-i}, z) > 0$ and, therefore, $v_x^{(-)} > 0$. Hence, a decreasing equilibrium never exists if types are statistically independent.

If, on the other hand, there is no strategic effect, *i.e.*, $\overline{\pi_j} = \overline{\pi_z} = 0$, so that firms have local monopolies, we have $v_x^{(-)}(x,z) = \overline{\pi_i} = \pi_e > 0$. Hence, also in this case a decreasing equilibrium does not exist.

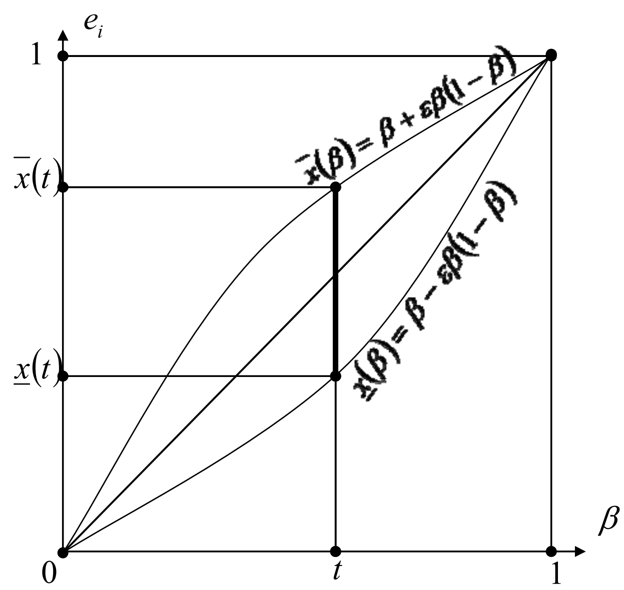

Figure 1. Support of the conditional distribution $F_e(x|t) = \Pr(e_j < x|\beta = t)$.

In the remaining of this section, we provide an example of market conditions where a unique symmetric bidding equilibrium exists that is decreasing. To this end, we take the following distribution F^* of firms' efficiency parameters. Let a macroeconomic fundamental (*e.g.*, interest rate, oil price, the growth rate or a state of the economy, *etc*.) β be distributed over the interval $[0, 1]$ in accordance with an arbitrary twice differentiable distribution function $F_\beta(t) \equiv \Pr(\beta < t)$. Then, for any given β, let all e_i be independently and uniformly distributed over the interval $[\underline{x}(\beta), \bar{x}(\beta)]$, where $\underline{x}(\beta) \equiv \beta - \varepsilon\beta(1-\beta)$, $\bar{x}(\beta) \equiv \beta + \varepsilon\beta(1-\beta)$, and $\varepsilon \in (0,1)$ is a parameter, *i.e.*, let the conditional distribution $F_e(x|t) \equiv \Pr(e_j < x|\beta = t)$ be

$$F_e(x|t) = \frac{x - \underline{x}(t)}{\bar{x}(t) - \underline{x}(t)} \; for \; x \in [\underline{x}(t), \bar{x}(t)].$$

This distribution has the following property: for small values of ε, if a firm i has a type x, the distribution of other firms' types conditional on x is concentrated on a small neighborhood of x. Thus, all firms competing in the aftermarket have approximately the same type, the Nash equilibrium is almost symmetric, and a decreasing equilibrium bidding function can be analytically calculated in the limit when ε converges to zero.[16] Figure 1 shows the support $[\underline{x}(t), \bar{x}(t)]$ of the conditional (uniform) distribution $F_e(x|t)$.

[16] This does not imply that ε must be very small for our example to work.

Proposition 4. *Let* $N = n+1$ *firms with constant marginal costs* $c - e_i > 0$ *compete in an auction for n licenses, let the winning firms compete in quantities in a market with constantly elastic demand* $Q = p^{-r}$*, and let firms' efficiency parameters* e_i *be distributed in accordance with the distribution* F^**. If the price elasticity r satisfies*

$$\frac{1}{n} < r < \bar{r}(n) \equiv \frac{n^2 + n + 2}{3n^2 + 3n - 2},$$

then there exists an $\tilde{\varepsilon}(r,n) > 0$ *so that for all* $\varepsilon \in (0, \tilde{\varepsilon})$ *the auction stage has a unique symmetric bidding equilibrium that is decreasing.*

In the example considered in Proposition 4, the demand elasticity r must not be too small ($r > 1/n$) in order to ensure that the aftermarket Nash equilibrium exists and is stable. On the other hand, r must be small enough ($r < \bar{r}(n)$) so that the strategic effect is sufficiently strong. The minimum number of licenses for this decreasing equilibrium to exists is $n = 3$, and the minimum number of competing firms is $N = n + 1 = 4$. The main reason is that the strategic effect should be strong enough and this effect gets stronger, the larger the number of firms competing in the market place.[17]

The condition of Proposition 4 is not as restrictive as it may seem at first sight. For example, in case of $n = 3$ licenses, the elasticity of individual demand, which in an (almost) symmetric equilibrium is n times higher than the market demand elasticity, must be such that $r_i \approx 3r \in (1,\ 1.235)$, whereas in case of $n = 6$ licenses, it must be $r_i \approx 6r \in (1,\ 2.129)$. Moreover, as we have shown in examples 1 and 2, convexity of firms' production costs strengthens the strategic effect. So, the condition of Proposition 4 can be further relaxed if firms have convex cost functions.

It is well known that if firms compete a la Cournot with demand being of constant elasticity, a proportional increase in firms' costs results in an increase in firms' profits. We exploit this property in constructing the decreasing equilibrium. However, iso-elastic demand alone is not sufficient: the *ex-ante* affiliation of firms' efficiency parameters has to be sufficiently strong for the equilibrium to exist.

Proposition 4 shows that even when the auction is efficient in that the most profitable firms win the licenses, the overall market allocation might be inefficient. The

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[17] Formally, the proposition is stated for the case where the number of interested firms is just one more than the number of available licenses. This is a technical requirement as otherwise we cannot integrate out the expressions in the profit function.

reason is that if firms' types are *ex-ante* affiliated, the strategic effect can be such that auction efficiency requires the least cost-efficient firms to win a license.

7. Discussion and Conclusion

In this article, we have shown that when multiple licenses are auctioned to firms which compete in an aftermarket, these licenses do not have to end up in the hands of the most efficient firms. This implies that auctions may create aftermarket inefficiencies. The main reasons for this result are the presence of an informational externality and the fact that rational bidders take a kind of adverse selection or winner's curse into account. The winner's curse that is present in our context is that firms prefer to compete with less efficient firms and that when the auction selects the most efficient firms, bidding firms have to take this selection effect into account. We have identified conditions under which efficient firms downsize their bid so much more than less efficient firms that an efficient selection equilibrium does not exist.

The model developed in this paper does not fit into the now standard assumptions of the affiliated valuation model (Milgrom and Weber, 1982). An important assumption in the affiliated valuation model is that a player's valuation is an increasing function of his own signal *as well as* of the private signals received by all other players. In our case, where firms receive a signal of their cost parameter, firms' valuation is an increasing function of its own signal, but a decreasing function of the signals of other firms. Moreover, but less important, a firm only cares about the signals received by other *winning* firms.

In this paper, we have focused our attention only on a standard multi-unit uniform-price auction. It can be shown, however, that other simultaneous-bid multi-unit auctions, *e.g.*, a pay-your-own-bid auction, also suffer from the same effects provided the negative externality or the *ex-ante* affiliation of players' types is sufficiently strong. If both the negative externality and the *ex-ante* affiliation are strong, a decreasing equilibrium can be constructed in a way similar to the one constructed in Section 6.

The analysis is much more complicated in case of sequential auctions, where licenses are sold one-by-one. It is easy to see that the last license ends up in the hands of the most efficient remaining firm. Nevertheless, the strategic effect might create inefficient market allocations in selling preceding licenses.

We have not allowed for resale in this paper. Resale opens up the possibility that in case of an inefficient allocation of licenses, an efficient firm buys a license from a less efficient firm. Such a single transaction would also be mutually beneficial because for given competitors' efficiency levels, an efficient firm makes more profits than a less efficient firm. However, it is much less clear whether such a transaction is feasible in case other firms are also allowed to transact so that a sequential resale market would emerge. The result of Gomes and Jehiel (2005) suggests that allowing for resale can make the outcome even worse: the absence of "negative externality-free" allocations in our model may lead to inefficient allocation.

Apart from the fact that reselling is sometimes not allowed or prohibitively costly, there is another good reason not to consider the possibility of reselling. For example, in case a decreasing equilibrium exists, n most efficient firms together make less profit than n least efficient firms do. If, together with reselling, we allow firms to make side payments to other firms for not selling their licenses, one can show that the license holders (together) can "outbid" an offer of a more efficient firm as the profits they would lose when this new firm competes in the market are larger than the profit the newcomer could make. Hence, a decreasing bidding equilibrium yields an *ex-post* efficient allocation from the perspective of the *coalition* of winning firms.

This paper does not consider the question whether a mechanism exists in the present situation that yields a market efficient outcome. This is an interesting, but non-trivial, question. What is clear, however, is that the results obtained by Ausubel (2004), and a much more general result of Perry and Reny (2002) do not apply here, as the present model does not satisfy the standard monotonicity assumptions of the affiliated valuation model (as indicated above).

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Appendix

Proof of Proposition 1. Let us define Z to be the type of the firm that submits the n^{th} highest bid amongst all $(N - 1)$ firms other than firm i, *i.e.*, Z is the n^{th} highest order statistics among e_j, $j \neq i$. We denote the distribution of Z conditional on $e_i = x$ by $G^{(+)}(z|x) = \Pr(Z < z|e_i = x)$, and the corresponding density function by $g^{(+)}(z|x)$.

Suppose that all firms other than i follow the bidding function $b^{(+)}(e)$, and Z takes a value z. We consider a firm i, which has a cost parameter $e_i = x$ and which bids $b^{(+)}(y)$. If $y < z$, firm i loses the auction and receives no profit. If, on the other hand, $y > z$, firm

i gets a license at the auction price $w = b^{(+)}(z)$, which yields the conditional expected profit $v^{(+)}(x,z) - b^{(+)}(z)$ to firm i. The unconditional expected profit of a firm with cost parameter x and a bid $b^{(+)}(y)$ is

$$V^{(+)}(x,y) = \int\limits_{z<y} \left(v^{(+)}(x,z) - b^{(+)}(z)\right) g^{(+)}(z|x)\,dz\,.$$

Maximizing $V^{(+)}(x,y)$ with respect to y yields the first-order condition $x = \arg\max_y V^{(+)}(x,y)$, *i.e.*, $b^{(+)}(x) = v^{(+)}(x,x)$. This bidding function $b^{(+)}(x)$ is an increasing function only if $v_x^{(+)}(x,x) + v_z^{(+)}(x,x) \geq 0$ The necessary second-order condition in this case can be written as $v_x^{(+)}(x,x) \geq 0$.

Suppose now that $v_x^{(+)} > 0$ and $v_x^{(+)}(x,x) + v_z^{(+)}(x,x) > 0$. In order to check that the increasing function $b^{(+)}(x) = v^{(+)}(x,x)$ is indeed an optimal bid, we evaluate $V^{(+)}(x,x) - V^{(+)}(x,y)$ for any $y \neq x$:

$$\begin{aligned} V^{(+)}(x,x) - V^{(+)}(x,y) &= \int\limits_{y<z<x} \left(v^{(+)}(x,z) - v^{(+)}(z,z)\right) g^{(+)}(z|x)\,dz \\ &= \int_y^x \left(\int_z^x v_x^{(+)}(t,z)\,dt \right) g^{(+)}(z|x)\,dz \geq 0 \end{aligned}$$

This shows that firm i has no profitable deviations. ■

Proof of Lemma 1. If a firm i gets a license, has a type $e_i = x$, and chooses s_i, whereas all its competitors have the same type z, the market profit of firm i can be written as follows:

$$E\left(\pi(s_i, s_{-i}, e_i) \middle| e_i = x, e_k = z\right).$$

Maximizing this expression with respect to s_i yields the following first-order condition:

$$0 = E\left(\pi_i(s_i^*, s_{-i}^*, e_i) \middle| e_i = x, e_k = z\right) = \pi_i\left(s(x,z), s_{-i}^*, x\right).$$

Differentiating it with respect to x and z, taking into account that firm i is of type x and all other firms are of type z and choose $s_k^* = s(z,z)$, and evaluating the resulting expression at $z = x$ yields:

$$0 = \pi_{i,i}\frac{\partial s}{\partial e} + \pi_{i,e} \text{ and}$$

$$0 = \pi_{i,i}\frac{\partial s}{\partial z} + (n-1)\pi_{i,j}\left(\frac{\partial s}{\partial e} + \frac{\partial s}{\partial z}\right).$$

Solving them together provides us with the following partials of the Nash equilibrium strategy $s(x,z)$ at $z = x$:

$$\frac{\partial s}{\partial e} = -\frac{\pi_{i,e}}{\pi_{i,i}} \text{ and}$$

$$\frac{\partial s}{\partial z} = \frac{(n-1)\pi_{i,j}\pi_{i,e}}{(\pi_{i,i} + (n-1)\pi_{i,j})\pi_{i,i}}.$$

Then, substituting these expressions into $\overline{\pi_j} = (\partial s / \partial e)\pi_j$ and $\overline{\pi_z} = (n-1)(\partial s / \partial z)\pi_j$ finally yields

$$\overline{\pi_i}(x,x,x) = \pi_e > 0,$$

$$\overline{\pi_j}(x,x,x) = -\frac{\pi_j \pi_{i,e}}{\pi_{i,i}} < 0 \text{ and}$$

$$\overline{\pi_z}(x,x,x) = \frac{(n-1)^2 \pi_j \pi_{i,j} \pi_{i,e}}{(\pi_{i,i} + (n-1)\pi_{i,j})\pi_{i,i}}.$$

In quantity competition settings it is $\pi_j < 0$ and $\pi_{i,e} > 0$ whereas in price competition for substitutes it is $\pi_j > 0$ and $\pi_{i,e} < 0$. Therefore, $\overline{\pi_z}(x,x,x) > 0$ when s are strategic substitutes ($\pi_{i,j} > 0$), and $\overline{\pi_z}(x,x,x) < 0$ when s are strategic complements ($\pi_{i,j} < 0$). ■

Proof of Proposition 2. We will show that under the condition of the proposition, the necessary existence condition $v_x^{(+)}(e,e) + v_z^{(+)}(e,e) \geq 0$ for an increasing equilibrium fails at $e = \bar{e}$. To this end, we first calculate firms' aftermarket Nash equilibrium strategy $s(\bar{e},\bar{e})$ and its partial derivatives. Then, we calculate firms' valuation function $v^{(+)}(\bar{e},\bar{e})$ and its partial derivatives.

In the limit when $(x,z) \to (\bar{e},\bar{e})$, the Nash equilibrium strategy $s(x,z)$ can be written as $s(x,z) = s^{(0)} - (\bar{e} - x)s^{(1)} - (\bar{e} - z)s^{(2)}$, where

$$s^{(0)} \equiv s(\bar{e},\bar{e}),$$

$$s^{(1)} \equiv \frac{\partial s}{\partial x}(\bar{e},\bar{e}), \text{ and}$$

$$s^{(2)} \equiv \frac{\partial s}{\partial z}(\bar{e},\bar{e}).$$

Dropping arguments in all functions evaluated at $(x,z)=(\bar{e},\bar{e})$, the first-order condition $0=E\left(\pi_i\left(s_i^*,s_{-i}^*,e_i\right)\middle|e_i=x,e_l\le z<e_k\right)$ for independent types in the first-order approximation becomes

$$\begin{aligned}0&=E\left(\pi_i-\left(\left(\bar{e}-x\right)s^{(1)}+\left(\bar{e}-z\right)s^{(2)}\right)\pi_{i,i}-\left(\bar{e}-x\right)\pi_{i,e}\middle|z<e_k\right)-\\&\qquad-(n-1)\pi_{i,j}E\left(\left(\bar{e}-e_k\right)s^{(1)}+\left(\bar{e}-z\right)s^{(2)}\middle|z<e_k\right)\\&=\pi_i-\left(\pi_{i,i}s^{(1)}+\pi_{i,e}\right)\left(\bar{e}-x\right)-\\&\qquad-\left(\pi_{i,i}s^{(2)}+(n-1)\pi_{i,j}\left(s^{(2)}+\lim_{z\uparrow\bar{e}}\frac{E\left(\bar{e}-e_k\middle|z<e_k\right)}{\bar{e}-z}s^{(1)}\right)\right)\left(\bar{e}-z\right)\end{aligned}$$

At $(x,z)=(\bar{e},\bar{e})$, the first-order condition implies $0=\pi_i$. Then, denoting

$$\tilde{E}\equiv\lim_{z\uparrow\bar{e}}\frac{E\left(\bar{e}-e_k\middle|z<e_k\right)}{\bar{e}-z}=\lim_{z\uparrow\bar{e}}\frac{\int_z^{\bar{e}}\left(\bar{e}-x\right)f(x)dx}{\left(\bar{e}-z\right)\int_z^{\bar{e}}f(x)dx}=\lim_{z\uparrow\bar{e}}\frac{\left(\bar{e}-z\right)h(z)}{1+\left(\bar{e}-z\right)h(z)}=\frac{\gamma}{\gamma+1},$$

the first-order condition implies the following two equations, which determine $s^{(1)}$ and $s^{(2)}$:

$$\begin{cases}0=\pi_{i,i}s^{(1)}+\pi_{i,e}\\0=\pi_{i,i}s^{(2)}+(n-1)\pi_{i,j}\left(s^{(2)}+\tilde{E}s^{(1)}\right)\end{cases}.$$

Solving this system yields

$$\begin{cases}s^{(1)}=-\dfrac{\pi_{i,e}}{\pi_{i,i}}\\s^{(2)}=\dfrac{(n-1)\pi_{i,j}\pi_{i,e}}{\pi_{i,i}\left(\pi_{i,i}+(n-1)\pi_{i,j}\right)}\tilde{E}\end{cases}$$

Hence, firms' valuation function $v^{(+)}(x,z)$ in the limit $(x,z)=(\bar{e},\bar{e})$ can be written as

$$\begin{aligned}v^{(+)}(x,z)&=E\left(\pi-(n-1)\left(\left(\bar{e}-e_k\right)s^{(1)}+\left(\bar{e}-z\right)s^{(2)}\right)\pi_j-\left(\bar{e}-x\right)\pi_e\middle|z<e_k\right)\\&=\pi-(n-1)\left(\tilde{E}s^{(1)}+s^{(2)}\right)\pi_j\left(\bar{e}-z\right)-\left(\bar{e}-x\right)\pi_e\\&=\pi+\frac{(n-1)\pi_j\pi_{i,e}\tilde{E}}{\left(\pi_{i,i}+(n-1)\pi_{i,j}\right)}\left(\bar{e}-z\right)-\left(\bar{e}-x\right)\pi_e\end{aligned}$$

Finally, the necessary condition $v_x^{(+)}(e,e)+v_z^{(+)}(e,e)\ge0$ for an increasing equilibrium to exist can be written at $e=\bar{e}$ as

$$0\le v_x^{(+)}\left(\bar{e},\bar{e}\right)+v_z^{(+)}\left(\bar{e},\bar{e}\right)=\pi_e-\frac{(n-1)\pi_j\pi_{i,e}\tilde{E}}{\left(\pi_{i,i}+(n-1)\pi_{i,j}\right)}=\pi_e-\frac{\gamma}{\gamma+1}\frac{(n-1)\pi_j\pi_{i,e}}{\left(\pi_{i,i}+(n-1)\pi_{i,j}\right)}.$$

In accordance with Lemma 1, $\pi_j \pi_{i,e} < 0$. Hence, necessary condition fails when

$$\frac{\pi_{i,e}}{\pi_e} \frac{\gamma}{\gamma+1} \frac{(n-1)\pi_j}{\left(\pi_{i,i} + (n-1)\pi_{i,j}\right)} > 1 .$$

Therefore, the above inequality is a sufficient condition for an increasing equilibrium not to exist. This ends the proof. ■

Proof of Proposition 4. First, we derive firms' aftermarket Nash equilibrium strategy, which we denote by $s(x, z; \varepsilon)$ to emphasize its dependence on ε, under the assumption that in the auction stage all they follow an increasing bidding function $b^{(+)}(e)$. Under the assumption that the profit function π is twice continuously differentiable, $s(x, z; \varepsilon)$ is continuously differentiable with respect to all its argument, and we represent it as a first-order Tailor expansion. We show that this bidding function $b^{(+)}(e)$ does not satisfy the second order condition, hence, an increasing symmetric bidding equilibrium does not exist. Second, we repeat the previous exercise for a decreasing bidding function $b^{(-)}(e)$ and analyze conditions under which $b^{(-)}(e)$ is indeed an equilibrium bidding function.

From now on we denote by x a type of a firm i. Types of all other (winning and losing) firms are e_j, (e_k and $e_l = z$ respectively), and we define

$$\lambda_j \equiv (e_j - x)/\varepsilon ,$$

$$\underline{t}(x) \equiv \left((1+\varepsilon) - \sqrt{(1+\varepsilon)^2 - 4\varepsilon x} \right) / (2\varepsilon) \text{ and}$$

$$\bar{t}(x) \equiv \left(\sqrt{(1-\varepsilon)^2 + 4\varepsilon x} - (1-\varepsilon) \right) / (2\varepsilon),$$

so that for any $x \in [0,1]$:

$$x \equiv \underline{x}(\bar{t}(x)) \text{ and } x \equiv \bar{x}(\underline{t}(x)).$$

Thus, for given x, conditional distribution $F_\beta(t|x) \equiv \Pr(\beta < t | e_i = x)$ has the support $[\underline{t}(x), \bar{t}(x)]$, and conditional distribution $F_e(\lambda|x) \equiv \Pr(\lambda_j < \lambda | e_i = x)$ has the support $[\underline{\lambda}(x), \bar{\lambda}(x)]$, where both $\underline{\lambda}(x)$ and $\bar{\lambda}(x)$ are bounded:[18]

$$\underline{\lambda}(x) = (\underline{t} - x)/\varepsilon - \underline{t}(1 - \underline{t}) \geq -0.5 \text{ and}$$

$$\bar{\lambda}(x) = (\bar{t} - x)/\varepsilon + \bar{t}(1 - \bar{t}) \leq 0.5 .$$

[18] These inequalities can be obtained by minimizing $\underline{\lambda}(x)$ and maximizing $\bar{\lambda}(x)$ with respect to x.

Denoting $s(x,z;\varepsilon)$ itself and its partial derivatives evaluated at $(x,x;0)$ as

$$s^{(0)}(x) \equiv s(x,x;0),\ s^{(1)}(x) \equiv \frac{\partial s}{\partial x}(x,x;0),\ s^{(2)}(x) \equiv \frac{\partial s}{\partial z}(x,x;0), \text{ and } s^{(3)}(x) \equiv \frac{\partial s}{\partial \varepsilon}(x,x;0),$$

allows $s(x,z;\varepsilon)$ and $s(e_j,z;\varepsilon)$ to be written in the first-order approximation as

$$s(x,z;\varepsilon) = s^{(0)} + \left(s^{(2)}\lambda_z + s^{(3)}\right)\varepsilon \text{ and } s(e_j,z;\varepsilon) = s^{(0)} + \left(\lambda_j s^{(1)} + \lambda_z s^{(2)} + s^{(3)}\right)\varepsilon .$$

Dropping arguments lists in all functions evaluated at $z = x$, we write the first-order condition

$$0 = E\left(\pi_i\left(s(x,z;\varepsilon), s(e_1,z;\varepsilon), \ldots, e_i\right) \middle| e_i = x, e_l = z < e_k\right)$$

as follows:

$$\begin{aligned} 0 &= E\left(\pi_i\left(s^{(0)} + \left(s^{(2)}\lambda_z + s^{(3)}\right)\varepsilon, s^{(0)} + \left(\lambda_1 s^{(1)} + \lambda_z s^{(2)} + s^{(3)}\right)\varepsilon, \ldots, x\right) \middle| e_i = x, e_l = z < e_k\right) \\ &= E\left(\pi_i + \varepsilon\left(\pi_{i,i}\left(s^{(2)}\lambda_z + s^{(3)}\right) + \pi_{i,j}\left(s^{(1)}\sum\nolimits_k \lambda_k + (n-1)\left(s^{(2)}\lambda_z + s^{(3)}\right)\right)\right) \middle| e_i = x, e_l = z < e_k\right) \\ &= \pi_i + \left(\left(\pi_{i,i} + (n-1)\pi_{i,j}\right)\left(s^{(2)}\lambda_z + s^{(3)}\right) + (n-1)\pi_{i,j} s^{(1)} \lim_{\varepsilon \to 0} E\left(\lambda_k \middle| e_i = x, e_l = z < e_k\right)\right)\varepsilon \end{aligned}$$

Let, first, $\varepsilon = 0$. In this case, the first-order condition implies $0 = \pi_i$, which for a given profit function

$$\pi(s_i, s_{-i}, e_i) = s_i\left(\left(\sum\nolimits_j s_j\right)^{-1/r} - c + e_i\right)$$

yields the symmetric aftermarket Nash equilibrium strategy:

$$s(x,x;0) = s^{(0)} = \frac{1}{n}\left(\frac{(nr-1)}{nr(c-x)}\right)^r .$$

In this case

$$\pi = \frac{(c-x)s^{(0)}}{(nr-1)},\ \pi_j = -\frac{(c-x)}{(nr-1)},\ \pi_{i,i} = -\frac{\left(2nr - (1+r)\right)(c-x)}{(nr-1)nrs^{(0)}},$$

$$\pi_{i,j} = -\frac{\left(nr - (1+r)\right)(c-x)}{(nr-1)nrs^{(0)}}, \text{ and } s^{(1)} + s^{(2)} = \frac{rs^{(0)}}{(c-x)}.$$

Using $0 = \pi_i$, we rewrite the first-order condition for $\varepsilon \neq 0$ and $\lambda_z = 0$ as follows:

$$0 = \left(\pi_{i,i} + (n-1)\pi_{i,j}\right)s^{(3)} + (n-1)\pi_{i,j} s^{(1)} \lim_{\varepsilon \to 0} \frac{E\left(e_k \middle| e_i = x, e_l = z < e_k\right) - x}{\varepsilon},$$

so that

$$s^{(3)} = -\frac{(n-1)\pi_{i,j}}{\left(\pi_{i,i} + (n-1)\pi_{i,j}\right)} s^{(1)} \lim_{\varepsilon \to 0} \frac{H^{(+)}(x,x) - x}{\varepsilon},$$

where we defined functions

$$H^{(\pm)}(x,z) \equiv E\left(e_k \middle| e_i = x, e_l = z, \pm z < \pm e_k\right).$$

Properties of $H^{(\pm)}(x,z)$ are derived in the following lemma, which is proven after the proof of the proposition.

Lemma 2. Let e_i be distributed in accordance with the distribution function F^*. Then, $H^{(\pm)}(x,z)$ for small ε can be written as follows:

$$H^{(\pm)}(x,z) = \frac{nx + (n+2)z}{2(n+1)} \pm \frac{nx(1-x)}{(n+1)}\varepsilon + o(\varepsilon),$$

Using Lemma 2, we rewrite $s^{(3)}$ as follows:

$$s^{(3)} = -\frac{n(n-1)\pi_{i,j}x(1-x)}{(n+1)(\pi_{i,i} + (n-1)\pi_{i,j})}s^{(1)}$$

Plugging this expression into the first-order condition yields:

$$\begin{aligned} s^{(2)} &= -\frac{(n-1)\pi_{i,j}}{(n+1)(\pi_{i,i} + (n-1)\pi_{i,j})}s^{(1)} \lim_{\varepsilon \to 0} \frac{(n+1)(E(e_k | e_i = x, e_l = z < e_k) - x) - nx(1-x)\varepsilon}{(z-x)} \\ &= -\frac{(n+2)(n-1)\pi_{i,j}}{2(n+1)(\pi_{i,i} + (n-1)\pi_{i,j})}s^{(1)} \end{aligned}$$

Solving $s^{(1)} + s^{(2)} = rs^{(0)}/(c-x)$ together with the above expressions for $s^{(2)}$ and $s^{(3)}$ finally yields:

$$s^{(1)} = \frac{2r(n+1)(\pi_{i,i} + (n-1)\pi_{i,j})}{(c-x)(2(n+1)\pi_{i,i} + n(n-1)\pi_{i,j})}s^{(0)},$$

$$s^{(2)} = -\frac{r(n+2)(n-1)\pi_{i,j}}{(c-x)(2(n+1)\pi_{i,i} + n(n-1)\pi_{i,j})}s^{(0)}, \text{ and}$$

$$s^{(3)} = -\frac{2rn(n-1)\pi_{i,j}x(1-x)}{(c-x)(2(n+1)\pi_{i,i} + n(n-1)\pi_{i,j})}s^{(0)}.$$

Hence, the aftermarket Nash equilibrium strategy $s(x,z;\varepsilon)$ for small ε and $(z-x)$ is

$$s(x,z;\varepsilon) = \left(1 - \frac{r(n-1)\pi_{i,j}((n+2)(z-x) + 2nx(1-x)\varepsilon)}{(c-x)(2(n+1)\pi_{i,i} + n(n-1)\pi_{i,j})}\right)s^{(0)}.$$

This ends the analysis of the aftermarket stage of the game.

In order show that an increasing symmetric bidding equilibrium does not exist for small ε, we calculate firms' valuation function $v^{(+)}(x,z;\varepsilon)$, and verify that for given parameters' restrictions, $v_x^{(+)}(x,x;0) < 0$. Using the first-order approximation for

$s(x,z;\varepsilon)$, firms' valuation function $v^{(+)}(x,z;\varepsilon)$ in the first-order approximation can be written as follows:

$$
\begin{aligned}
v^{(+)} &= E\left(\pi\left(s(x,z;\varepsilon), s(e_1,z;\varepsilon), z\right) \middle| e_i = x, e_l \le z < e_k\right) \\
&= E\left(\pi\left(s^{(0)} + \left(s^{(2)}\lambda_z + s^{(3)}\right)\varepsilon, s^{(0)} + \left(\lambda_1 s^{(1)} + \lambda_z s^{(2)} + s^{(3)}\right)\varepsilon, \ldots, x\right) \middle| e_i = x, e_l = z < e_k\right) \\
&= E\left(\pi + \pi_j \sum\nolimits_k \left(\lambda_k s^{(1)} + \lambda_z s^{(2)} + s^{(3)}\right)\varepsilon \middle| e_i = x, e_l = z < e_k\right) \\
&= \pi + (n-1)\pi_j\left(\left(\lim_{\varepsilon\to 0}\frac{\left(H^{(+)}(x,z) - x\right)s^{(1)}}{\varepsilon} + s^{(3)}\right)\varepsilon + s^{(2)}(z-x)\right) \\
&= \pi + \frac{r(n-1)\pi_j\pi_{i,i}\left((n+2)(z-x) + 2nx(1-x)\varepsilon\right)}{(c-x)\left(2(n+1)\pi_{i,i} + n(n-1)\pi_{i,j}\right)} s^{(0)}
\end{aligned}
$$

Substituting expressions for π, π_j, $\pi_{i,i}$, $\pi_{i,j}$, and $s^{(0)}$ finally yields the following valuation function $v^{(+)}(x,z;\varepsilon)$:

$$
v^{(+)} = \frac{1}{n^2 r}\left(\frac{nr(c-x)}{(nr-1)}\right)^{1-r}\left(1 - \frac{r(n-1)(2nr-(1+r))\left((n+2)(z-x) + 2nx(1-x)\varepsilon\right)}{(c-x)\left(2(n+1)(2nr-(1+r)) + n(n-1)(nr-(1+r))\right)}\right)
$$

But then

$$
v_x^{(+)}(x,x;0) = -\left(\frac{nr(c-x)}{(nr-1)}\right)^{1-r}\frac{(nr-1)\left((n^2+n+2) - (3n^2+3n-2)r\right)}{n^2(c-x)\left(\left(r(n^2+2n+3)+1\right)(nr-1) + (1-r)(1+2r)\right)} < 0,
$$

provided $r < \bar{r}(n)$. This implies that $v_x^{(+)}(x,x;\varepsilon) < 0$ for sufficiently small (but strictly positive) ε, so that an increasing symmetric bidding equilibrium does not exist.

Suppose now that there exists a symmetric decreasing equilibrium bidding function $b^{(-)}(e)$. In a similar way as above, firms' valuation function $v^{(-)}(x,z;\varepsilon)$ can be written as follows:

$$
v^{(-)} = \frac{1}{n^2 r}\left(\frac{nr(c-x)}{(nr-1)}\right)^{1-r}\left(1 - \frac{r(n-1)(2nr-(1+r))\left((n+2)(z-x) - 2nx(1-x)\varepsilon\right)}{(c-x)\left(2(n+1)(2nr-(1+r)) + n(n-1)(nr-(1+r))\right)}\right).
$$

But then

$$
v_x^{(-)}(x,x;0) = -\left(\frac{nr(c-x)}{(nr-1)}\right)^{1-r}\frac{(nr-1)\left((n^2+n+2) - (3n^2+3n-2)r\right)}{n^2(c-x)\left(\left(r(n^2+2n+3)+1\right)(nr-1) + (1-r)(1+2r)\right)} < 0,
$$

for $r < \bar{r}(n)$, and

$$
v_z^{(-)}(x,x;0) = -\left(\frac{nr(c-x)}{(nr-1)}\right)^{1-r}\frac{(n+2)(n-1)(2nr-(1+r))}{n^2(c-x)\left(2(n+1)(2nr-(1+r)) + n(n-1)(nr-(1+r))\right)} < 0.
$$

By continuity argument, there exists an $\tilde{\varepsilon} > 0$ so that $v_x^{(-)}(x,z;\varepsilon) < 0$ and $v_z^{(-)}(x,z;\varepsilon) < 0$ for all $\varepsilon \in (0,\tilde{\varepsilon})$ and feasible z. Therefore, the proposed function $b^{(-)}(x) = v^{(-)}(x,x)$ is indeed decreasing and is a unique symmetric equilibrium bidding function. ■

Proof of Lemma 2. We denote distribution functions as follows:

$F_\beta(t) \equiv \Pr(\beta < t)$,

$F_e(x,z) \equiv \Pr(e_i < x, e_k > z)$, and

$F_e(x,z|t) \equiv \Pr(e_i < x, e_k > z | \beta = t)$,

and the corresponding densities be

$f_\beta(t) \equiv dF_\beta(t)/dx = 1$,

$f_e(x,z) \equiv dF_e(x,z)/dx$, and

$f_e(x,z|t) \equiv dF_e(x,z|t)/dx$.

As $e_i \sim U(\underline{x}(\beta),\bar{x}(\beta))$, it follows that for $x \in [\underline{x}(t),\bar{x}(t)]$:

$$F_e(x,z|t) = \begin{cases} \dfrac{(x-\underline{x}(t))(\bar{x}(t)-z)^{n-1}}{(\bar{x}(t)-\underline{x}(t))^n}, & \text{if } z \in (\underline{x}(t),\bar{x}(t)) \\ \dfrac{(x-\underline{x}(t))}{\bar{x}(t)-\underline{x}(t)}, & \text{if } z \in (0,\underline{x}(t)) \end{cases}$$

And, therefore,

$$f_e(x,z|t) = \begin{cases} \dfrac{(\bar{x}(t)-z)^{n-1}}{(\bar{x}(t)-\underline{x}(t))^n}, & \text{if } z \in (t-\varepsilon t(1-t),\bar{x}(t)) \\ \dfrac{1}{\bar{x}(t)-\underline{x}(t)}, & \text{if } z \in (0,t-\varepsilon t(1-t)) \end{cases}$$

Hence,

$$\begin{aligned} f_\beta(t|x,z) &\equiv \frac{f_e(x,z|t)f_\beta(t)}{f_e(x,z)} \\ &= \begin{cases} \dfrac{(\bar{x}(t)-z)^{n-1} f_\beta(t)}{(\bar{x}(t)-\underline{x}(t))^n f_e(x,z)}, & \text{if } t \in (\max(\underline{t}(x),\underline{t}(z)),\min(\bar{t}(x),\bar{t}(z))) \\ \dfrac{f_\beta(t)}{(\bar{x}(t)-\underline{x}(t))f_e(x,z)}, & \text{if } t \in (\min(\bar{t}(x),\bar{t}(z)),\bar{t}(x)) \end{cases} \end{aligned}$$

where $\underline{t}$ and $\bar{t}$ are as defined in Proof of Proposition 4.

We define $\tilde{H}^{(+)}(x,z,t) \equiv E(e_k | e_i = x, e_l = z < e_k, \beta = t)$ and consider two cases.

a) When $z \geq x$, $\widetilde{H}^{(+)}(x,z,t) = \frac{1}{2}(\bar{x}(t) + z)$ for $t \in (\underline{t}(z), \bar{t}(x))$. Hence, $H^{(+)}(x,z)$ can be written as

$$H^{(+)}(x,z) = E_\beta\left(\widetilde{H}^{(+)}(x,z,\beta) \middle| e_i = x, e_l = z < e_k\right) = P_1(x,z)/Q_1(x,z), \text{ where}$$

$$P_1(x,z) \equiv f_e(x,z) \int_{\underline{t}(z)}^{\bar{t}(x)} \frac{1}{2}(\bar{x}(t) + z) f_\beta(t|x,z) dt = \int_{\underline{t}(z)}^{\bar{t}(x)} \frac{(\bar{x}(t) + z)(\bar{x}(t) - z)^{n-1} f_\beta(t)}{2(\bar{x}(t) - \underline{x}(t))^n} dt,$$

and

$$Q_1(x,z) \equiv f_e(x,z) \int_{\underline{t}(z)}^{\bar{t}(x)} f_\beta(t|x,z) dt = \int_{\underline{t}(z)}^{\bar{t}(x)} \frac{(\bar{x}(t) - z)^{n-1} f_\beta(t)}{(\bar{x}(t) - \underline{x}(t))^n} dt.$$

b) When $z \leq x$, $\widetilde{H}^{(+)}(x,z,t) = \frac{1}{2}(\bar{x}(t) + z)$ for $t \in (\underline{t}(x), \bar{t}(z))$ and $\widetilde{H}^{(+)}(x,z,t) = t$ for $t \in (\bar{t}(z), \bar{t}(x))$. Hence, $H^{(+)}(x,z)$ can be written as

$$H^{(+)}(x,z) = E_\beta\left(\widetilde{H}^{(+)}(x,z,\beta) \middle| e_i = x, e_l = z < e_k\right) = P_2(x,z)/Q_2(x,z), \text{ where}$$

$$P_2(x,z) \equiv \int_{\underline{t}(x)}^{\bar{t}(z)} \frac{(\bar{x}(t) + z)(\bar{x}(t) - z)^{n-1} f_\beta(t)}{2(\bar{x}(t) - \underline{x}(t))^n} dt + \int_{\bar{t}(z)}^{\bar{t}(x)} \frac{t f_\beta(t)}{\bar{x}(t) - \underline{x}(t)} dt,$$

and

$$Q_2(x,z) = \int_{\underline{t}(x)}^{\bar{t}(z)} \frac{(\bar{x}(t) - z)^{n-1} f_\beta(t)}{(\bar{x}(t) - \underline{x}(t))^n} dt + \int_{\bar{t}(z)}^{\bar{t}(x)} \frac{f_\beta(t)}{\bar{x}(t) - \underline{x}(t)} dt.$$

In order to evaluate $H^{(+)}(x,z)$ and its partials for small values of ε we use the 3rd-order approximation $\sqrt{1+\varepsilon} = 1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \frac{1}{16}\varepsilon^3 + o(\varepsilon^3)$, so that

$$\sqrt{(1-\varepsilon)^2 + 4\varepsilon x} = 1 + (2x-1)\varepsilon + 2x(1-x)\varepsilon^2 + 2x(1-x)(1-2x)\varepsilon^3 + o(\varepsilon^3), \text{ and}$$

$$\sqrt{(1+\varepsilon)^2 - 4\varepsilon x} = 1 - (2x-1)\varepsilon + 2x(1-x)\varepsilon^2 - 2x(1-x)(1-2x)\varepsilon^3 + o(\varepsilon^3).$$

Hence,

$$\bar{t}(x) = x + x(1-x)\varepsilon + x(1-x)(1-2x)\varepsilon^2 + o(\varepsilon^2),\ \bar{t}'(x) = 1 + o(1),$$

$$\underline{t}(x) = x - x(1-x)\varepsilon + x(1-x)(1-2x)\varepsilon^2 + o(\varepsilon^2), \text{ and } \underline{t}'(x) = 1 + o(1),$$

The uniform convergence with respect to $\varepsilon \in (0, \bar{\varepsilon}) \subset (0,1)$ of the limits

$$\lim_{x\to 0} \frac{\bar{t}(x)}{x} = \lim_{x\to 1} \frac{1-\underline{t}(x)}{1-x} = \frac{1}{(1-\varepsilon)}, \text{ and } \lim_{x\to 0} \frac{\underline{t}(x)}{x} = \lim_{x\to 1} \frac{1-\bar{t}(x)}{1-x} = \frac{1}{(1+\varepsilon)},$$

implies the following expressions for $\underline{t}$ and $\bar{t}$:

$$\bar{t}(x) = x + x(1-x)\varepsilon(1 + (1-2x)\varepsilon + o(\varepsilon)), \text{ and}$$

$\underline{t}(x) = x - x(1-x)\varepsilon(1-(1-2x)\varepsilon + o(\varepsilon))$.

We consider cases $z \geq x$ and $z \leq x$ separately.

a) Let $z \geq x$. Using the above Tailor expansions yields

$$Q_1(x,z) = \frac{1}{n}\left(f_\beta(x) + \frac{n-1}{n+1}\left(f_\beta(x)(1-2x) + f'_\beta(x)x(1-x)\right)\varepsilon \right),$$

$$\frac{\partial Q_1}{\partial x}(x,x) = \frac{\overline{t}'(x)\left(f_\beta(x) + \left(f'_\beta(x)x(1-x) - f_\beta(x)(1-2x)\right)\varepsilon\right)}{2x(1-x)\varepsilon},$$

$$\frac{\partial Q_1}{\partial z}(x,x) = -\frac{1}{2x(1-x)\varepsilon}\left(f_\beta(x) + \frac{n-2}{n} f'_\beta(x)x(1-x)\varepsilon \right),$$

$$P_1(x,x) = xQ_1(x,x) + \frac{f_\beta(x)x(1-x)\varepsilon}{n+1},$$

$$\frac{\partial P_1}{\partial x}(x,x) = \frac{\overline{t}'(x)\left(f_\beta(x) + x\left(f_\beta(x) + f'_\beta(x)(1-x)\right)\varepsilon\right)}{2(1-x)\varepsilon}, \text{ and}$$

$$\frac{\partial P_1}{\partial z}(x,x) = \frac{Q_1(x,x)}{2} - \frac{1}{2(1-x)\varepsilon}\left(f_\beta(x) + \frac{(1-x)}{n}\left((n-1)f_\beta(x) + (n-2)f'_\beta(x)x\right)\varepsilon \right).$$

Hence,

$$H^{(+)}(x,x+0) = x + \frac{n}{n+1}x(1-x)\varepsilon,$$

$$H_x^{(+)}(x,x+0) = \frac{n}{2(n+1)}, \text{ and } H_z^{(+)}(x,x+0) = \frac{(n+2)}{2(n+1)}.$$

b) Let $z \leq x$. As $P_2(x,x) = P_1(x,x)$ and $Q_2(x,x) = Q_1(x,x)$, it follows that $H^{(+)}(x,x-0) = H^{(+)}(x,x+0)$. Similarly, as $\partial P_2/\partial x = \partial P_1/\partial x$, $\partial P_2/\partial z = \partial P_1/\partial z$, $\partial Q_2/\partial x = \partial Q_1/\partial x$, and $\partial Q_2/\partial z = \partial Q_1/\partial z$ at (x,x), it follows that $H_x^{(+)}(x,x-0) = H_x^{(+)}(x,x+0)$ and $H_z^{(+)}(x,x-0) = H_z^{(+)}(x,x+0)$.

Thus at (x,x), $H^{(+)}(x,z)$ is continuously differentiable with the partials $H_x^{(+)}(x,x) = n/(2(n+1)) + o(1)$ and $H_z^{(+)}(x,x) = (n+2)/(2(n+1)) + o(1)$ and, therefore, can be written as

$$H^{(+)}(x,z) = \frac{nx + (n+2)z}{2(n+1)} + \frac{nx(1-x)}{(n+1)}\varepsilon + o(\varepsilon).$$

The expression for $H^{(-)}$ immediately follows from $H^{(-)}(x,z) = 1 - H^{(+)}(1-x,1-z)$. ■