# Lecture 2: The Continuous-Time Overlapping-Generations Model: Basic Theory and Applications

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# Outline

# Aims of this lecture (1)

- Study "work-horse" model of modern macroeconomics which is based on overlapping generations. Motivation for doing this:
  - Ricardian equivalence may be inappropriate (the chain of bequests may not be fully operational).
  - Tractable way to introduce (and study consequences of) heterogeneous agents.
  - Contains Ramsey model as a special case.
- Show some applications of the Blanchard-Yaari model.
  - Fiscal policy (crowding out effects of public consumption).
  - Debt neutrality revisited.

# Aims of this lecture (2)

- Extend the BY model in a number of minor/major directions:
  - Embed it in an endogenous growth model (how do a country's demography and economic growth interact?)
  - Age-dependent productivity (mimic life-cycle; reintroduces possibility of dynamic inefficiency – oversaving?).
  - Apply model to the small open economy (well-defined dynamics for the current account and consumption).
  - Endogenous labour supply (distorting aspects of taxation).
  - Life-cycle labour supply and retirement (ageing and retirement).
- Punchlines.

# Yaari's lessons (1)

- *Key questions studied by Yaari*:
  - How does a household behave if it faces *lifetime uncertainty*?
  - What kind of institutions exist to insure oneself against risk of death?
- Up to now we have only studied models without lifetime uncertainty:
  - In the two-period consumption model the agent knows he/she will expire at the end of period 2 (certain death).
  - In the Ramsey model the agent has an infinite horizon (certain eternal life).

# Yaari's lessons (2)

- A more realistic scenario:
  - Agent has a finite life.
  - Date of death is uncertain (but demographic data exist).
- Model complications: if date of death is uncertain then...
  - Complication (A): The agent faces a stochastic decision problem. Hence, the *expected utility hypothesis* must be used.
  - Complication (B): The restriction on terminal assets becomes more complicated. If $A(D)$ is real assets at time $D$ and $D$ is the (stochastic) time of death, then the terminal condition is that $A(D) \geq 0$ with probability one.

# Complication (A) solved by Yaari (1)

- Even though $D$ is stochastic we have a good idea about the distribution of $D$ in the population (ask the demographers). See **Figures 16.1 – 16.2**. The probability density function (PDF) of $D$ is:

$$\phi(D) \geq 0, \ \forall D \geq 0, \ \ \Phi(\bar{D}) = \int_0^{\bar{D}} \phi(D) dD = 1 \qquad \text{(S1)}$$

  - Densities are non-negative.
  - $D$ is non-negative.
  - $\bar{D}$ is the maximum lifetime.
- Define (stochastic) lifetime utility as:

$$\Lambda(D) \equiv \int_0^{D} U\left(C(\tau)\right) e^{-\rho\tau} d\tau \qquad \text{(S2)}$$

# Complication (A) solved by Yaari (2)

- But since $D$ is stochastic, an agent has the following objective function:

$$E(\Lambda(D)) \equiv \int_0^{\bar{D}} \phi(D)\Lambda(D)\,dD$$

- Using (S1) and (S2) we can derive a simple expression for expected lifetime utility:

$$\begin{aligned} E(\Lambda(D)) &\equiv \int_0^{\bar{D}} \phi(D)\left[\int_0^{D} U(C(\tau))\,e^{-\rho\tau}d\tau\right]dD \\ &= \int_0^{\bar{D}} \left[\int_\tau^{\bar{D}} \phi(D)dD\right] U(C(\tau))\,e^{-\rho\tau}d\tau \end{aligned}$$

# Complication (A) solved by Yaari (3)

- In compact form we write:

$$E(\Lambda(D)) \equiv \int_0^{\bar{D}} [1-\Phi(\tau)] \cdot U\left(C(\tau)\right) e^{-\rho\tau} d\tau \qquad \text{(S3)}$$

- In (S3), the term $1-\Phi(\tau)$ is the probability that the consumer will still be alive at time $\tau$:

$$1-\Phi(\tau) \equiv \int_\tau^{\bar{D}} \phi(D) dD$$

- The key thing to note about (S3) is that **lifetime uncertainty merely affects the rate at which felicity is discounted! This is Yaari's first lesson**

# Complication (B) solved by Yaari (1)

- Let's solve the next complication – dealing with the time-of-death wealth constraint.
- First he derives the appropriate terminal condition on real assets in the presence of lifetime uncertainty (but in the absence of insurance opportunities):

$$A(\bar{D}) = 0 \qquad \text{(S4)}$$

$$C(\tau) \leq w(\tau) \text{ whenever } A(\tau) = 0 \qquad \text{(S5)}$$

- (S4): Assets must be zero if agent reaches maximum age.
- (S5): If agent hits constraint in period $\tau$ then he/she must start saving ($\dot{A} > 0$) immediately to avoid defaulting.

# Complication (B) solved by Yaari (2)

- Second he shows that the consumption Euler equation is:

$$\frac{\dot{C}(\tau)}{C(\tau)} = \sigma\left(C(\tau)\right) \cdot \left[r(\tau) - \rho - \mu(\tau)\right] \tag{S6}$$

where $\mu(\tau)$ is the instantaneous probability of death at time $\tau$:

$$\mu(\tau) \equiv \frac{\phi(\tau)}{1 - \Phi(\tau)} \tag{S7}$$

**Note**: As we saw above, the lifetime uncertainty shows up as a heavier discounting of future felicity (one may not be around to enjoy felicity!). This is Yaari's first lesson again.

# Complication (B) solved by Yaari (3)

- Third, he argues that in reality all kind of insurance instruments exist. He introduces the so-called *actuarial note*.
  - Carries instantaneous yield $r^A(\tau)$.
  - If you buy €1 of such notes: yield of $r^A(\tau)$ while you are alive; you lose the principal when you die; yield must be higher than yield on other instruments $(r^A > r)$ **ANNUITY**.
  - If you sell such a note: get €1 from life insurance company; pay premium of $r^A$ while you are alive; debt is cancelled when you die; premium must compensate risk of the LIC $(r^A > r)$ **LIFE-INSURED LOAN**.

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# Complication (B) solved by Yaari (4)

- Under *actuarial fairness* the rate of return on the two types of instruments satisfy a no-arbitrage condition:

$$r^A(\tau) = r(\tau) + \mu(\tau) \tag{S8}$$

The yield on actuarial notes equals the interest rate on traditional assets plus the instantaneous probability of death.

- Fourth, Yaari shows that the household will always fully insure, i.e. will hold real wealth in the form of actuarial notes. This means that...
  - The budget identity is:

$$\dot{A}(\tau) = r^A(\tau)A(\tau) + w(\tau) - C(\tau)$$

  - The terminal asset condition is trivially met (WHY?):
  - The consumption Euler equation is:

$$\frac{\dot{C}(\tau)}{C(\tau)} = \sigma\left[C(\tau)\right] \cdot \left[r^A(\tau) - \rho - \mu(\tau)\right] \tag{S9}$$

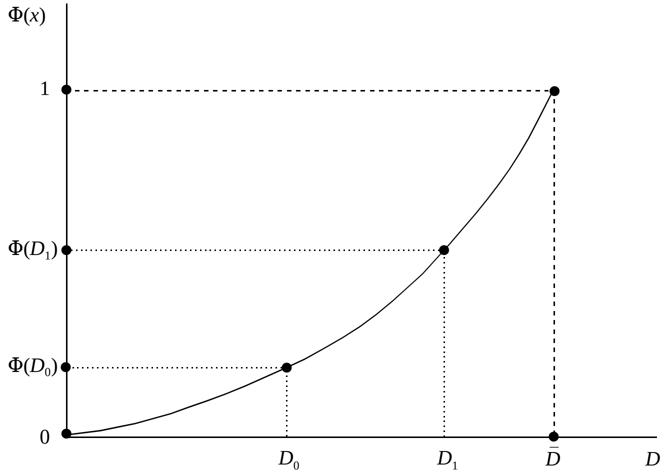
# Complication (B) solved by Yaari (5)

- Fifth, combining (S8) and (S9) we derive **Yaari's second lesson**:

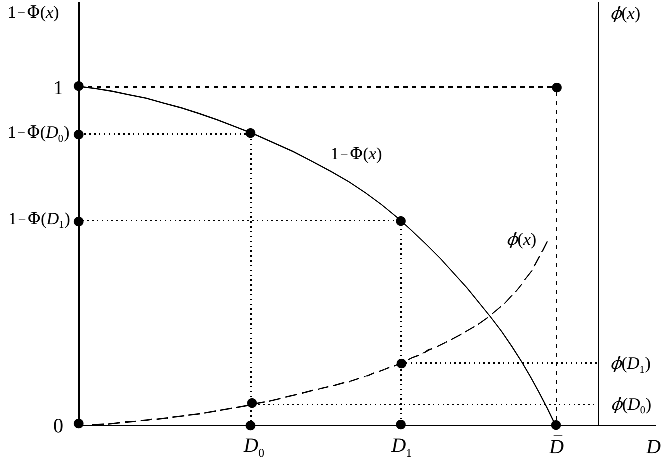
$$\frac{\dot{C}(\tau)}{C(\tau)} = \sigma\left(C(\tau)\right) \cdot \left[r(\tau) - \rho\right] \tag{S10}$$

With fully insured (actuarially fair) lifetime uncertainty, the death rate drops out of the consumption Euler equation altogether! (**Note**: The level of consumption is affected by the death rate.)

# Figure 16.1: Cumulative distribution function



# Figure 16.2: Density function and survival probability



# Visualization using Dutch demographic data (1)

- Use specific functional form for the demographic process (for $0 \leq u \leq \bar{D} \equiv \frac{\ln \mu_0}{\mu_1}$):

$$\Phi(u) \equiv \frac{e^{\mu_1 u} - 1}{\mu_0 - 1}, \qquad 1 - \Phi(u) \equiv \frac{\mu_0 - e^{\mu_1 u}}{\mu_0 - 1} \qquad \text{(S11)}$$

- Estimate parameters $\mu_0$ and $\mu_1$ using actual demographic data (Netherlands cohort born in 1920): $\hat{\mu}_0 = 41.06$ and $\hat{\mu}_1 = 0.0429$
- Estimated maximum age is $\bar{D} = 86.6$ years
- Life expectancy at birth of $65.4$ years.

# Visualization using Dutch demographic data (2)

- Recall (S11):

$$\Phi(u) \equiv \frac{e^{\mu_1 u} - 1}{\mu_0 - 1}, \qquad 1 - \Phi(u) \equiv \frac{\mu_0 - e^{\mu_1 u}}{\mu_0 - 1}$$

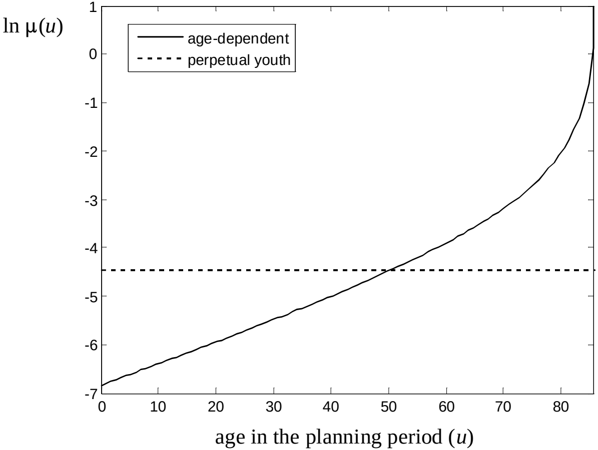
- From this expression we find (for $0 < u < \bar{D}$):

$$\phi(u) \equiv \frac{d\Phi(u)}{du} = \frac{\mu_1 e^{\mu_1 u}}{\mu_0 - 1} \qquad \text{(S12)}$$

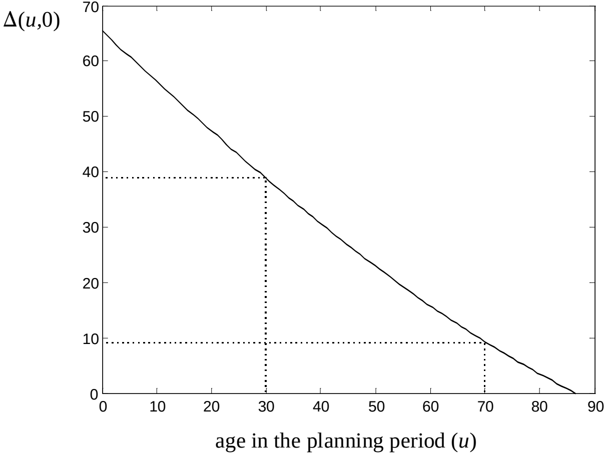
$$\mu(u) \equiv \frac{\phi(u)}{1 - \Phi(u)} = \frac{\mu_1 e^{\mu_1 u}}{\mu_0 - e^{\mu_1 u}} \qquad \text{(S13)}$$

- See **Figures 16.3 – 16.4**.

# Figure 16.3: Logarithm of the instantaneous mortality rate



# Figure 16.4: Expected remaining lifetime



# With actuarially fair (perfect) annuities

- annuity rate facing age-$u$ person is $r + \mu(u)$
- consumption growth is $\dot{C}(u)/C(u) = r - \rho > 0$
- consumption and assets over the life cycle:

$$\frac{C(u)}{w} = \frac{\Delta(0,r)}{\Delta(0,\rho)} e^{(r-\rho)u} \quad \text{(S14)}$$

$$\frac{A(u)}{w} = e^{(r-\rho)u} \frac{\Delta(0,r)}{\Delta(0,\rho)} \Delta(u,\rho) - \Delta(u,r) \quad \text{(S15)}$$

- demographic discount function:

$$\Delta(u,\psi) \equiv \frac{e^{\psi u}}{\mu_0 - e^{\mu_1 u}} \cdot \left[ \mu_0 \cdot \frac{e^{-\psi u} - e^{-\psi \bar{D}}}{\psi} + \frac{e^{(\mu_1 - \psi)u} - e^{(\mu_1 - \psi)\bar{D}}}{\mu_1 - \psi} \right] \quad \text{(S16)}$$

- See Figures 16.5 – 16.6.

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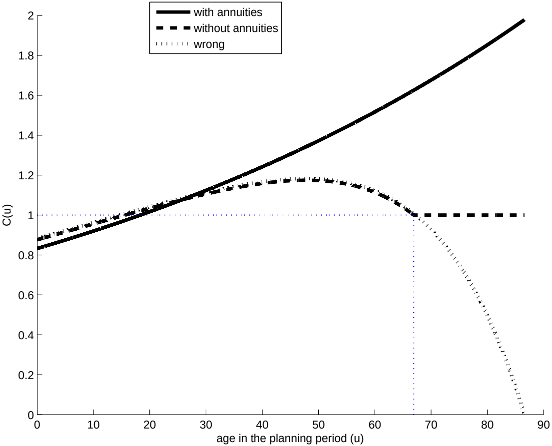
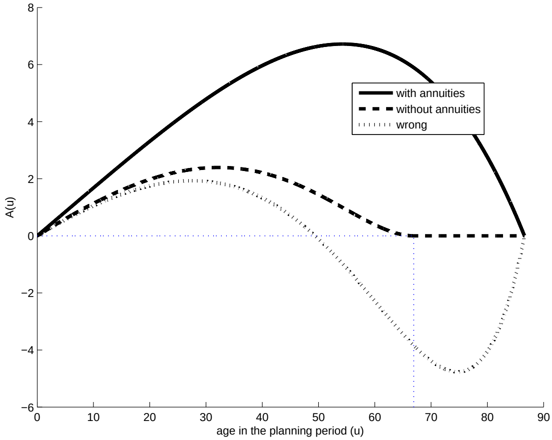
# Figure 16.5: Consumption



# Figure 16.6: Financial assets



# In the absence of annuities

- individual faces time-of-death borrowing constraint, $A(u) \geq 0$
- consumption growth is $\dot{C}(u)/C(u) = r - (\rho + \mu(u))$ until borrowing constraint is encountered
- individual runs out of financial assets and consumes wage income thereafter
- See **Figures 16.5 – 16.6**.

# Bird's-eye view (1)

- Blanchard (1985): general equilibrium model with finite lives and overlapping generations
- *Key idea*: Blanchard embedded Yaari's approach in a general equilibrium framework. He simplified the approach by assuming that the planning horizon is *age-independent* and is distributed *exponentially* ("perpetual youth" assumption).
- Implications of this assumption:
  - The death rate equals $\mu$ (a constant, independent of age).
  - The expected planning horizon equals $1/\mu$ in that case. (**Note:** As $\mu = 0$ we have the Ramsey model again.)
  - Household decision rules linear in age parameter (see below).

# Bird's-eye view (2)

- He furthermore assumed that at each instant a *large cohort* of agents in born (bare of any financial assets as they do not receive inheritance–unloved agents). Implications:
  - Denote the cohort born at time $\tau$ by $P(\tau,\tau) \equiv \mu P(\tau)$ (with $P(\tau)$ large): the first index is the birth date and the second index is time.
  - All agents face a probability of death of $\mu$ so $\mu P(\tau)$ agents die at each instant (#births equals #deaths so population size is constant and $P(\tau)$ can be normalized to unity).
  - With large cohorts "probabilities and frequencies coincide" and given the first assumption we can trace the size of each cohort over time:

$$\begin{aligned} P(v,\tau) &= P(v,v)e^{\mu(v-\tau)} \\ &= \mu e^{\mu(v-\tau)}, \quad \tau \geq v \end{aligned}$$

# Bird's-eye view (3)

- Because we know cohort sizes we can aggregate all surviving households (nice for macro model).
- Eventually, as people die off the cohorts vanish.
- We can now derive the implications for individual and aggregate household behaviour. Details are in the chapter. Sketch of the outcome here.

# Individual household behaviour (1)

- Expected lifetime utility of agent of cohort $v$ in period $t$:

$$E(\Lambda(v,t)) \equiv \int_t^{\infty} [1-\Phi(\tau-t)] \ln C(v,\tau) e^{\rho(t-\tau)} d\tau$$
$$= \int_t^{\infty} \ln C(v,\tau) e^{(\rho+\mu)(t-\tau)} d\tau$$

- Budget identity:

$$\dot{A}(v,\tau) = [r(\tau)+\mu] A(v,\tau) + w(\tau) - T(\tau) - C(v,\tau) \quad \text{(S18)}$$

- No Ponzi Game (NPG) condition:

$$\lim_{\tau\to\infty} e^{-R^A(t,\tau)} A(v,\tau) = 0, \quad R^A(t,\tau) \equiv \int_t^{\tau} [r(s)+\mu] ds$$

# Individual household behaviour (2)

- Decision rule for consumption:

$$C(v,t) = (\rho+\mu)\left[A(v,t)+H(t)\right] \quad \text{(S19)}$$

$$H(t) \equiv \int_t^{\infty} \left[w(\tau)-T(\tau)\right] e^{-R^A(t,\tau)} d\tau \quad \text{(S20)}$$

- Notes:
  - Marginal propensity to consume out of total wealth is $\rho+\mu$ (does not feature an age index due to the perpetual youth assumption).
  - Human wealth discounted at the annuity rate of interest, $r(\tau)+\mu$.

# Aggregate household behaviour (1)

- We know that the size of cohort $v$ at time $t$ is $\mu e^{\mu(v-t)}$. This means that we can define aggregate variables by aggregating over all existing agents at time $t$. For example, aggregate consumption is:

$$C(t) \equiv \mu \int_{-\infty}^{t} e^{\mu(v-t)} C(v,t) dv$$

- In view of (S19) *aggregate consumption* satisfies:

$$\begin{aligned} C(t) &\equiv \mu \int_{-\infty}^{t} e^{\mu(v-t)} (\rho+\mu) \left[ A(v,t) + H(t) \right] dv \\ &= (\rho+\mu) \Bigg[ \underbrace{\mu \int_{-\infty}^{t} e^{\mu(v-t)} A(v,t) dv}_{A(t)} + \underbrace{\mu \int_{-\infty}^{t} e^{\mu(v-t)} H(t) dv}_{H(t)} \Bigg] \\ &= (\rho+\mu) \left[ A(t) + H(t) \right] \end{aligned}$$

# Aggregate household behaviour (2)

- Similarly, the *aggregate budget identity* can be derived:

$$\dot{A}(t) = r(t)A(t) + w(t) - T(t) - C(t) \qquad \text{(S21)}$$

  The market rate of interest (**not** the annuity rate) features in the aggregate budget identity: the term $\mu A(t)$ is a transfer–via the life insurance companies–from agents who die to agents who stay alive.

- Recall (S18) (for period $t$):

$$\dot{A}(v,t) = [r(t) + \mu]\, A(v,t) + w(t) - T(t) - C(v,t)$$

# Aggregate household behaviour (3)

- The consumption Euler equation for individual agents is:

$$\frac{\dot{C}(v,t)}{C(v,t)} = r(t) - \rho$$

The "aggregate Euler equation" satisfies:

$$\begin{aligned} \frac{\dot{C}(t)}{C(t)} &= [r(t) - \rho] - \mu(\rho + \mu)\frac{A(t)}{C(t)} \\ &= \frac{\dot{C}(v,t)}{C(v,t)} - \mu\frac{C(t) - C(t,t)}{C(t)} \end{aligned}$$

- Note: Aggregate consumption growth differs from individual consumption growth due to the turnover of generations. Newborns are poorer than the average household and therefore drag down aggregate consumption growth.

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# Phase diagram of the Blanchard-Yaari model (1)

- We now have all the ingredients of the BY model (firm behaviour is standard; we allow for debt creation in the GBC): see **Table 16.1**.
- In **Figure 16.7** we show the phase diagram for a special case of the BY model, under the assumption that there is no government at all ($T(t) = G(t) = B(t) = 0$).
- The $\dot{K} = 0$ line represents $(C, K)$ combinations for which net investment is zero. It has the usual properties:
  - Golden rule point at $A_2$.
  - $\dot{K} > 0$ ($\dot{K} < 0$) for points below (above) the $\dot{K} = 0$ line (see horizontal arrows).

# Phase diagram of the Blanchard-Yaari model (2)

- The $\dot{C}=0$ line represents $(C,K)$ combinations for which *aggregate* consumption is constant over time. It has some unusual properties:
  - It lies entirely to the left of the dashed line, representing the Keynes-Ramsey capital stock (for which $r^{KR}=\rho$). Why? Using the aggregate Euler equation for the BY model we get:

$$\begin{aligned}\frac{\dot{C}(t)}{C(t)} &= [r(t)-\rho]-\mu(\rho+\mu)\frac{K(t)}{C(t)}=0 \quad \Rightarrow \\ r^{BY}-\rho &= \mu(\rho+\mu)\frac{K}{C}^{BY} \quad \Rightarrow \\ r^{BY} &> \rho\end{aligned}$$

  The interest rate strictly higher than $\rho$ (due to generational turnover). Hence, $K^{BY}$ strictly smaller than $K^{KR}$.

# Phase diagram of the Blanchard-Yaari model (3)

- Continued.
  - The $\dot{C} = 0$ line is upward sloping. Can be understood by comparing points $\mathrm{E}_0$, B, and C in **Figure 16.7**. In $\mathrm{E}_0$ and B $r$ is the same but $K/C$ is higher in B. To restore $\dot{C} = 0$ we must have a move to point C, where $K$ is lower than in B ($r$ higher) and $K/C$ is lower.
  - For points above (below) the $\dot{C} = 0$ line, the generational turnover effect is too low (too strong), and aggregate consumption growth is positive (negative). See the vertical arrows in Figure 16.7.
- The BY model without a government features a unique equilibrium at $\mathrm{E}_0$ which is saddle point stable.

# Table 16.1: The Blanchard-Yaari model

$$\dot{C}(t) = [r(t) - \rho]\, C(t) - \mu(\rho + \mu)\, [K(t) + B(t)] \qquad \text{(T1.1)}$$

$$\dot{K}(t) = Y(t) - C(t) - G(t) - \delta K(t) \qquad \text{(T1.2)}$$

$$\dot{B}(t) = r(t)B(t) + G(t) - T(t) \qquad \text{(T1.3)}$$

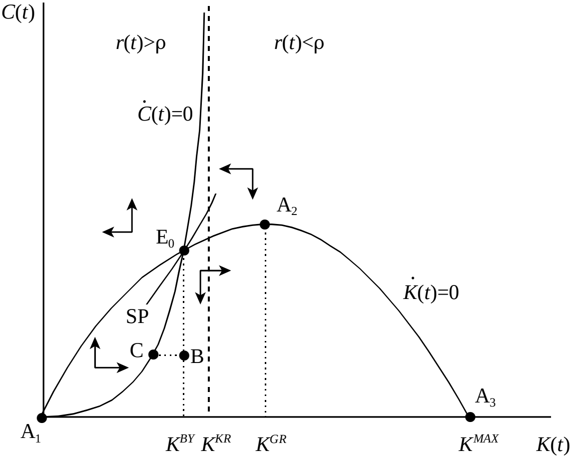
$$r(t) + \delta = F_K(K(t), L(t)) \qquad \text{(T1.4)}$$

$$w(t) = F_L(K(t), L(t)) \qquad \text{(T1.5)}$$

$$L(t) = 1 \qquad \text{(T1.6)}$$

$$Y(t) = F(K(t), L(t)) \qquad \text{(T1.7)}$$

# Figure 16.7: Phase diagram of the Blanchard-Yaari model



# Some basic model properties

- *Fiscal policy*: increase in government consumption financed by means of lump-sum taxes. Issues:
  - Crowding out of private by public consumption?
  - Intergenerational redistribution of resources? How does this work?
- *Non-neutrality of debt*.
  - Does government debt matter?
  - Do deficit-financed policies differ from balanced-budget policies?

# Fiscal policy (1)

- Unanticipated and permanent increase in $G$ financed by increase in $T$ (recall $T$ is the same for all agents, regardless of their vintage).
- Abstract from government debt: $\dot{B} = B = 0$ and GBC is static, $G = T$.
- The shock is analyzed in **Figure 16.8**.
  - The $\dot{K} = 0$ line shifts down by the amount of the shock.
  - The $\dot{C} = 0$ line is unchanged (no supply effect of tax).
  - Steady state shifts from $E_0$ to $E_1$: $C(\infty) \downarrow$ and $K(\infty) \downarrow$ (the latter does not occur in Ramsey model).

# Fiscal policy (2)

- Continued.
  - Transitional dynamics: jump from $E_0$ to A (at impact) followed by gradual move along saddle path from A to $E_1$ thereafter. (Recall: no t.d. in Ramsey model.)
  - Crowding out results:

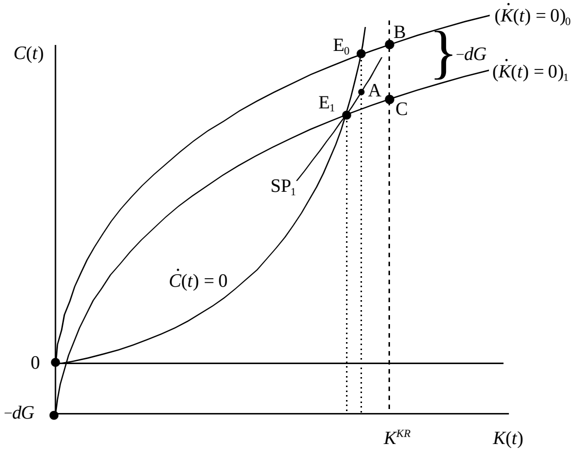
$$-1 < \frac{dC(0)}{dG} < 0$$
$$\frac{dC(\infty)}{dG} < -1$$

  Less than one-for-one at impact but more than one-for-one in the long run!

# Fiscal policy (3)

- Economic intuition: the $T \uparrow$ causes an intergenerational redistribution of resources away from future towards present generations.
    - At impact $C(v,0) \downarrow$ because $H(0) \downarrow$ (due to $T \uparrow$).
    - Households discount net labour income stream, $w - T$, by annuity rate $r + \mu$ (higher than market interest rate, $r$).
    - Hence, the drop in $C(v,0)$, $C(0)$, and $H(0)$ is not large enough, so that private investment in crowded out: $\dot{K}(0) \downarrow$.
    - Over time $K(t) \searrow$, so that $[w(t) - T] \searrow$, $r(t) \nearrow$, and $H(t) \searrow$.
    - Future newborns poorer than newborns in initial steady state (the former have less capital to work with).

# Figure 16.8: Fiscal policy in the Blanchard-Yaari model



# Non-neutrality of debt (1)

- The fact that $T$ causes intergenerational redistribution in the fiscal policy case hints at the non-neutrality of debt.
- Ricardian non-equivalence can be proven by looking at a simple accounting exercises: substitute the GBC into the HBC.
- The aggregate wealth constraint facing household features the following definition for total wealth:

$$\begin{aligned} A(t) + H(t) &\equiv K(t) + B(t) + H(t) \\ &= K(t) + B(t) + \int_t^\infty [w(\tau) - T(\tau)]\, e^{-R^A(t,\tau)} d\tau \\ &= K(t) + \int_t^\infty [w(\tau) - G(\tau)]\, e^{-R^A(t,\tau)} d\tau + \Omega(t) \end{aligned}$$

# Non-neutrality of debt (2)

- Here $\Omega(t)$ is defined as:

$$\Omega(t) \equiv B(t) - \int_t^\infty [T(\tau) - G(\tau)] \underbrace{e^{-R^A(t,\tau)}}_{(a)} d\tau \qquad \text{(S22)}$$

**Note:** Ricardian equivalence holds iff $\Omega(t) \equiv 0$!

- Recall that the GBC can be written as:

$$0 = B(t) - \int_t^\infty [T(\tau) - G(\tau)] \underbrace{e^{-R(t,\tau)}}_{(b)} d\tau \qquad \text{(S23)}$$

- In (S22) primary surpluses are discounted with the annuity rate (see (a)) whereas the market rate is used in (S23) (see (b)).
    - Hence, $\Omega(t)$ only vanishes iff the birth rate is zero, so that $R^A(t,\tau) = R(t,\tau)$, i.e. in the Ramsey model.
    - If $\mu > 0$ then $\Omega(t) \neq 0$ and Ricardian equivalence fails: the path of $T(\tau)$ and the initial debt level do not drop out of the aggregate HBC.