REFLECTIONS ON ARMAX SYSTEMS

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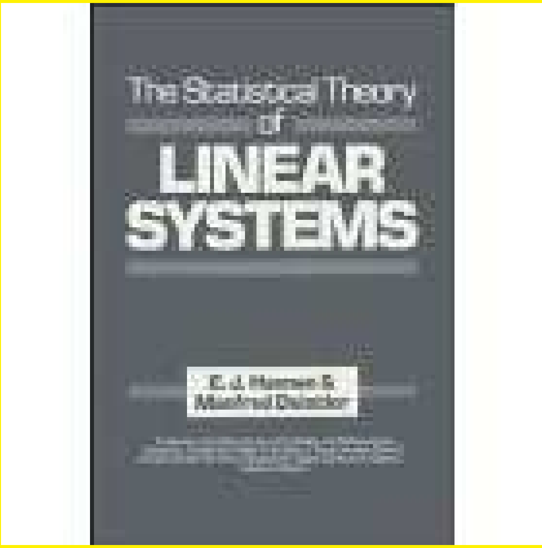
In honor of Manfred Deistler on the occasion of his retirement

ARMAX



ARMAX systems

$$\begin{aligned} A_0 y(t) + A_1 y(t+1) + \cdots + A_{\mathtt{L}_1} y(t+\mathtt{L}_1) &= X_0 u(t) + X_1 u(t+1) + \cdots + X_{\mathtt{L}_2} u(t+\mathtt{L}_2) \\ &+ M_0 \varepsilon(t) + M_1 \varepsilon(t+1) + \cdots + M_{\mathtt{L}_3} \varepsilon(t+\mathtt{L}_3) \end{aligned}$$

ARMAX systems

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+ M_0 \varepsilon(t) + M_1 \varepsilon(t+1) + \cdots + M_{\mathtt{L}_3} \varepsilon(t+\mathtt{L}_3)
\end{aligned}
$$

$$A(\sigma)y = X(\sigma)u + M(\sigma)\varepsilon$$

σ = **the shift,** $\quad \sigma f(t) := f(t+1)$

A, X, M: **real polynomial matrices**

ARMAX systems

$$A(\sigma)y = X(\sigma)u + M(\sigma)\varepsilon$$

$\sigma =$ **the shift,** $\quad \sigma f(t) := f(t+1)$

A, X, M**: real polynomial matrices**

$y, u : \mathbb{Z} \rightarrow \mathbb{R}^{\mathtt{p}}, \mathbb{R}^{\mathtt{m}}$**,** u **input,** y **output**

the variables whose dynamic relation is modeled

$\varepsilon : \mathbb{Z} \rightarrow \mathbb{R}^{\ell}$ **disturbances, 'noise'**

A**: AutoRegressive-part**

M**: Moving Average-part**

X**: EXogenous-part**

Equivalent model class

$$\sigma x = Ax + Bu + G\varepsilon, y = Cx + Du + J\varepsilon$$

$\sigma =$ **the shift,** $\quad \sigma f(t) := f(t+1)$

$y, u : \mathbb{Z} \to \mathbb{R}^{\mathtt{p}}, \mathbb{R}^{\mathtt{m}}$ u **input,** y **output**

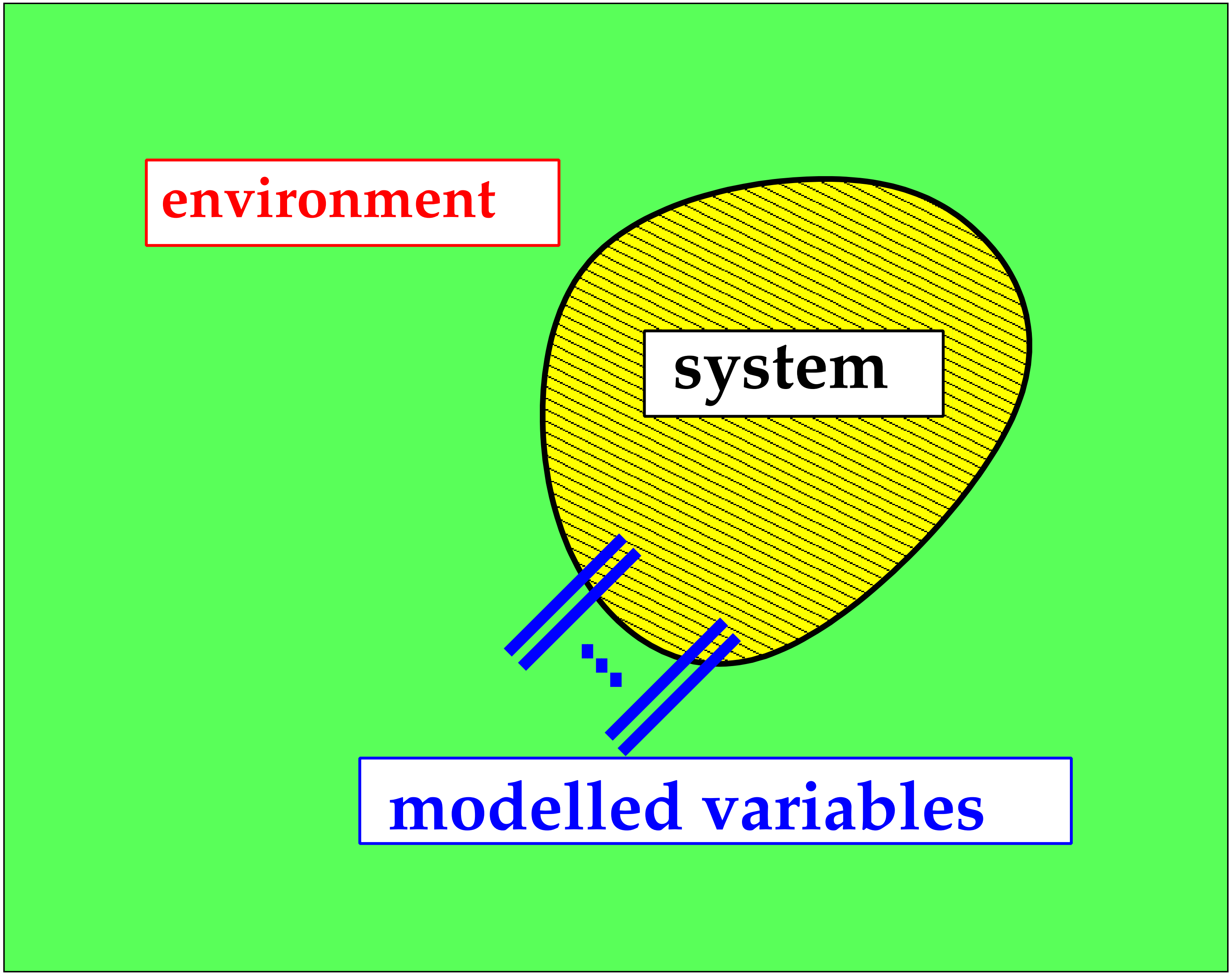
the variables whose dynamic behavior is modeled

$\varepsilon : \mathbb{Z} \to \mathbb{R}^{\ell}$ **disturbance, 'noise'**

$x : \mathbb{Z} \to \mathbb{R}^{\mathtt{n}}$ **auxiliary state variables**

A, B, C, G, D, J**: real matrices**

Modeling idea



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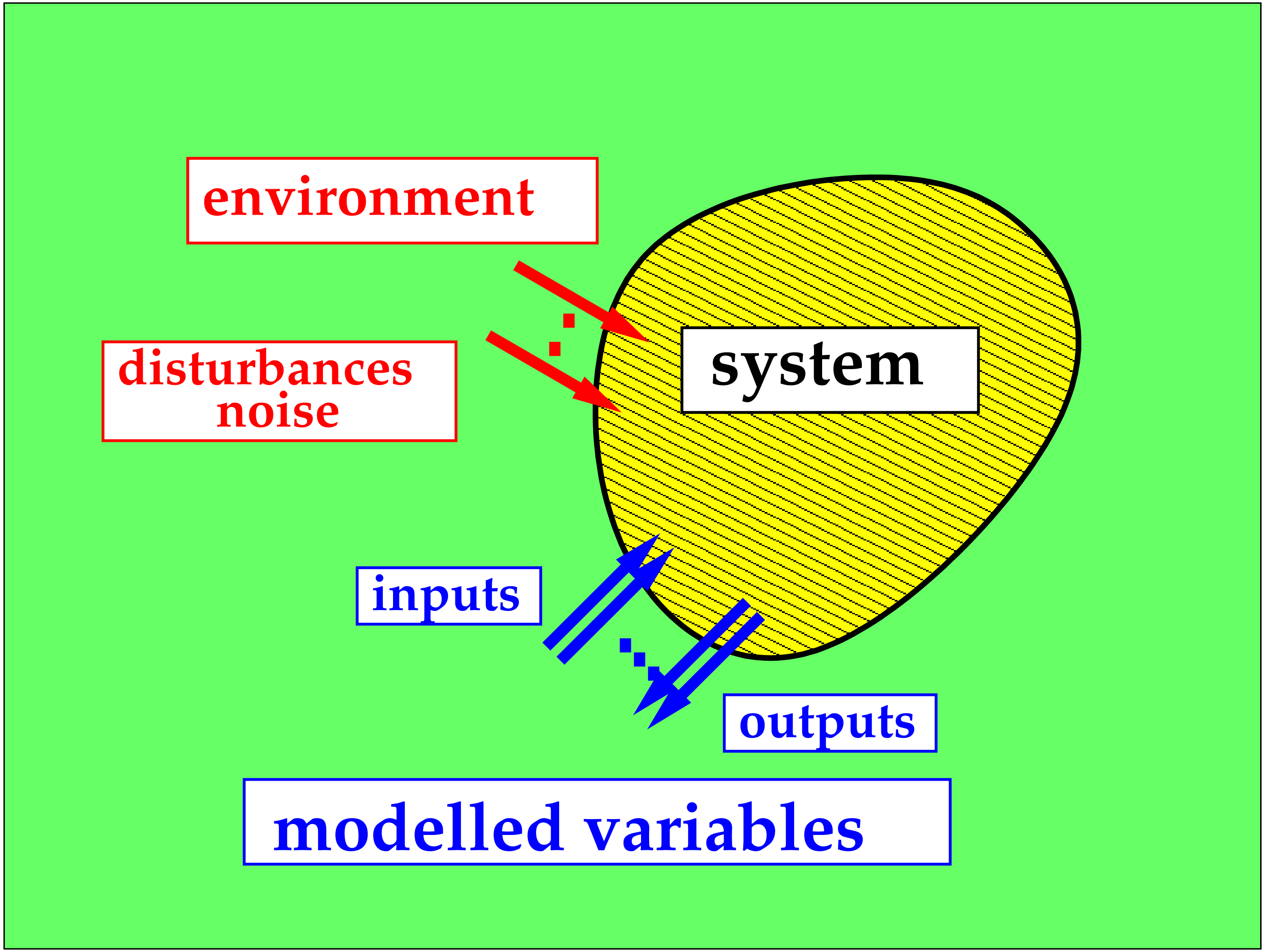

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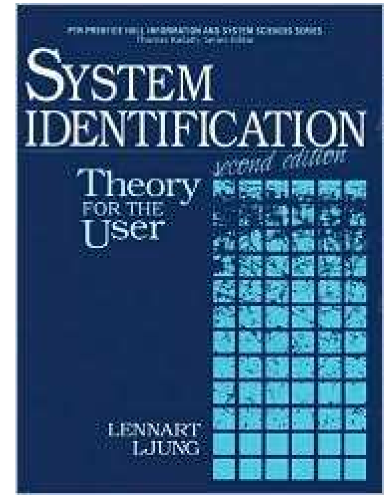
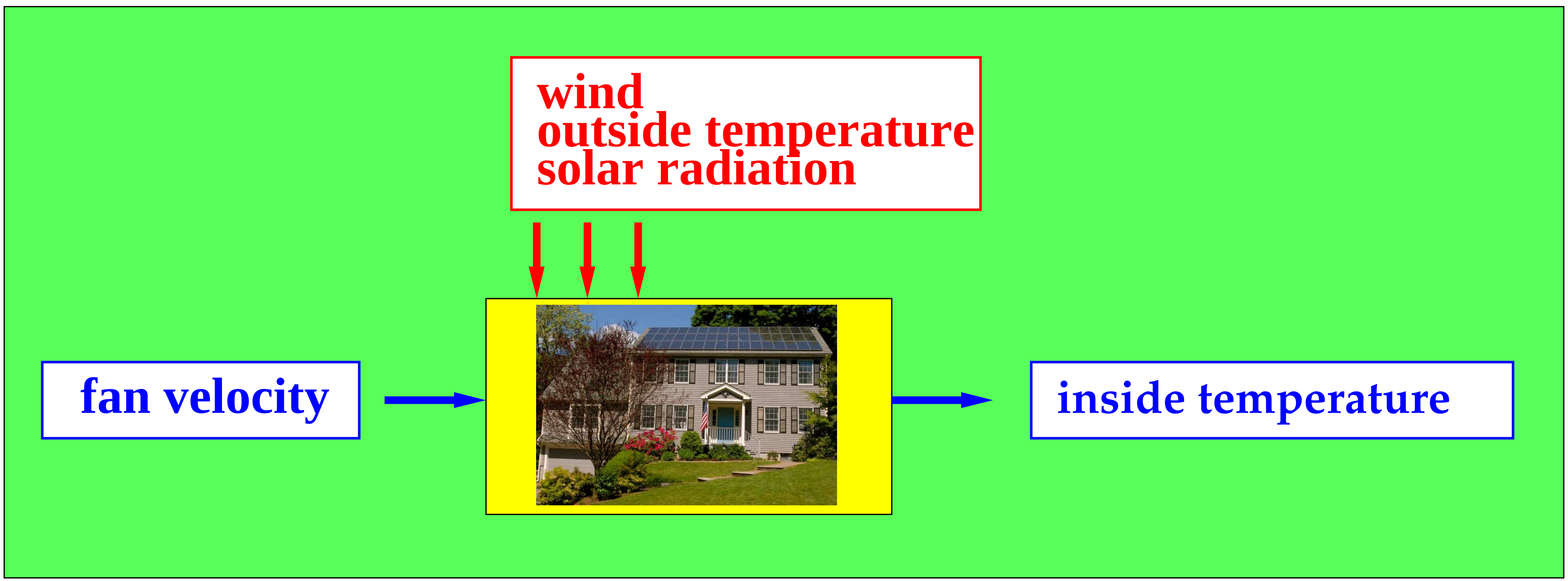
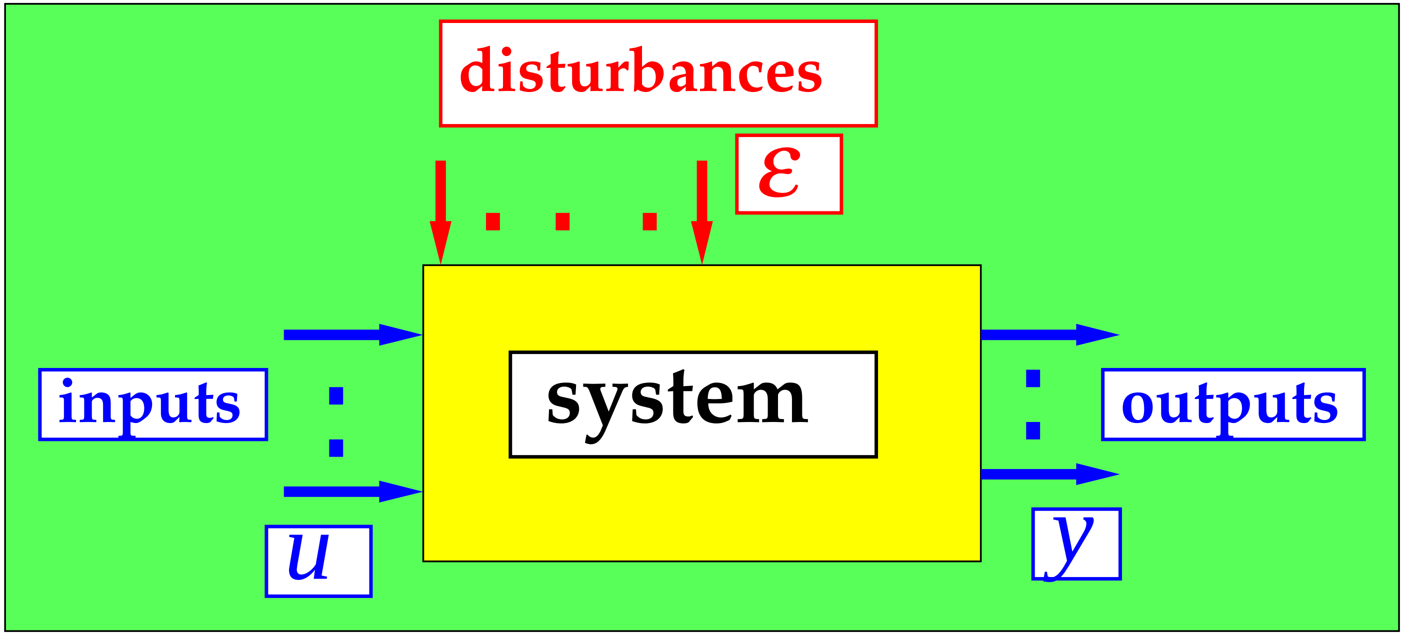


Example

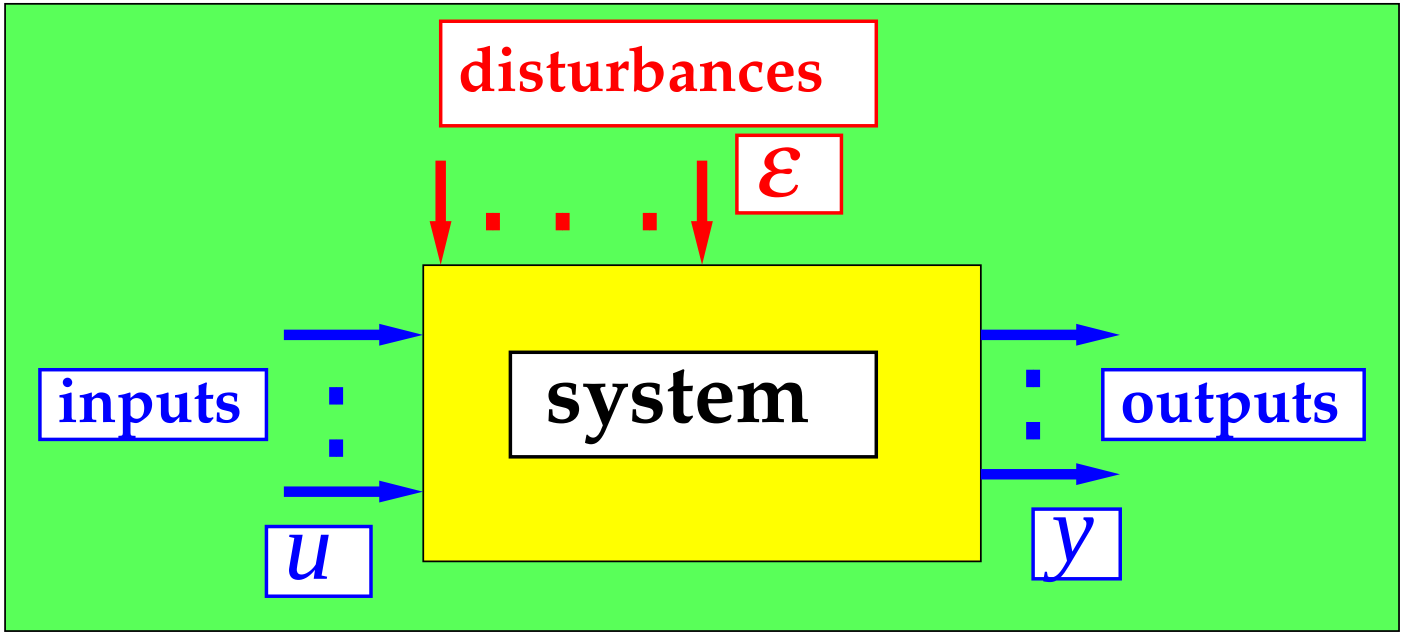


Inertia $\rightsquigarrow$ **difference equation with lags** $\Rightarrow$ **ARMAX**

Modeling idea



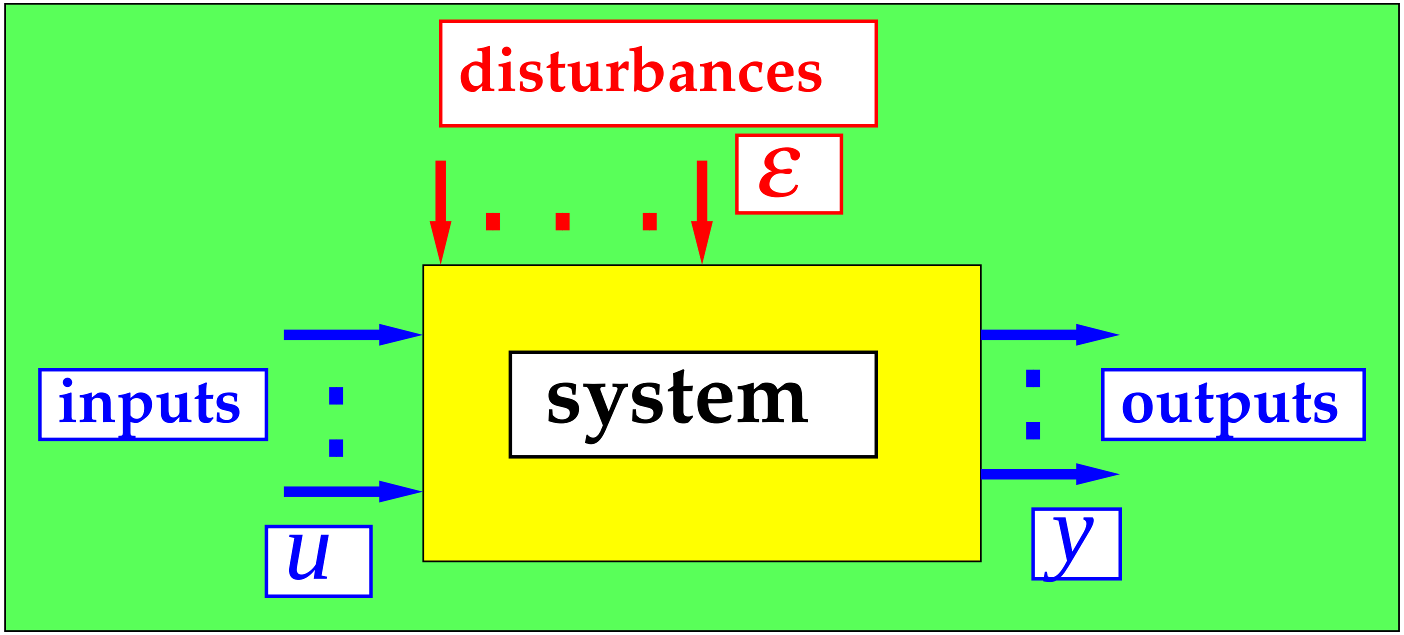
Modeling idea



Typical assumptions:

- ε a stationary stochastic (vector) process
- u a stochastic process, typically independent of ε
- suitable assumptions on A, M, X
- $\Rightarrow y$ stochastic process

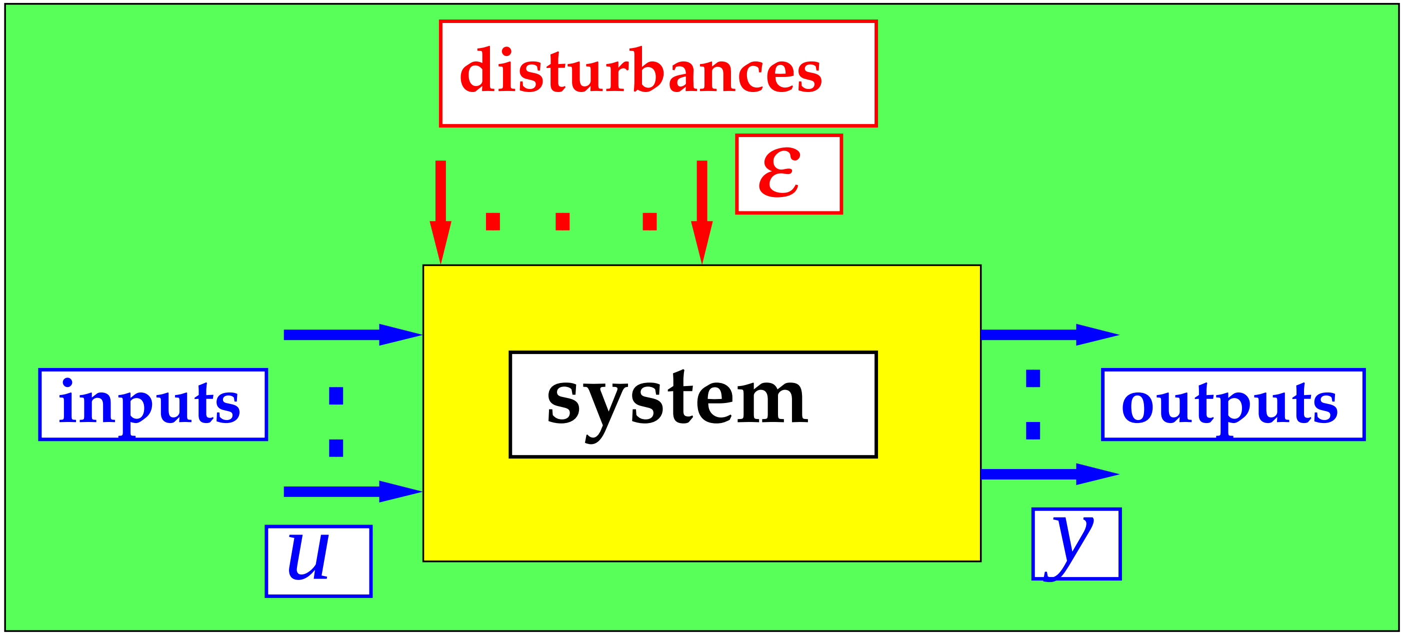
Modeling idea



Reflections:

- the separation of system variables into inputs and outputs
- the stochastic nature of disturbance inputs ε
- the input nature of external disturbances

Modeling idea



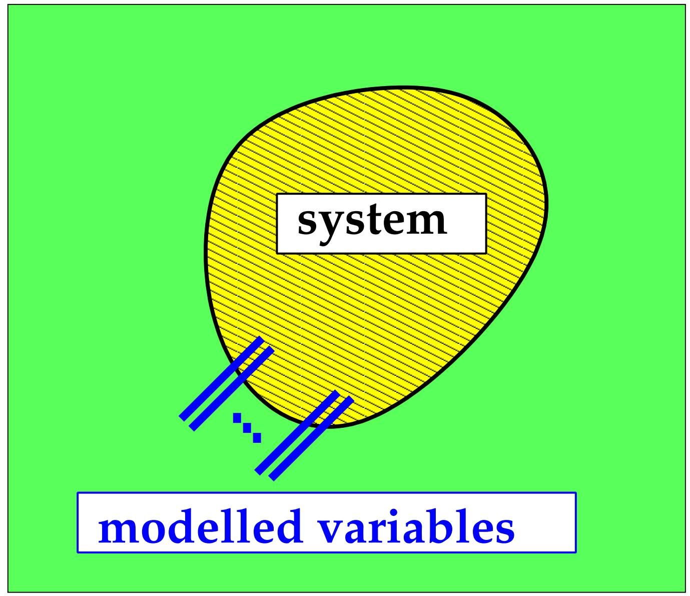
Reflections:

- **separation of system variables**

  into inputs u and outputs y
- **the stochastic nature of disturbance inputs ε**
- **the input nature of external disturbances**

INPUTS and OUTPUTS

Closed systems



If the system variables are completely generated 'internally', we speak of closed systems.

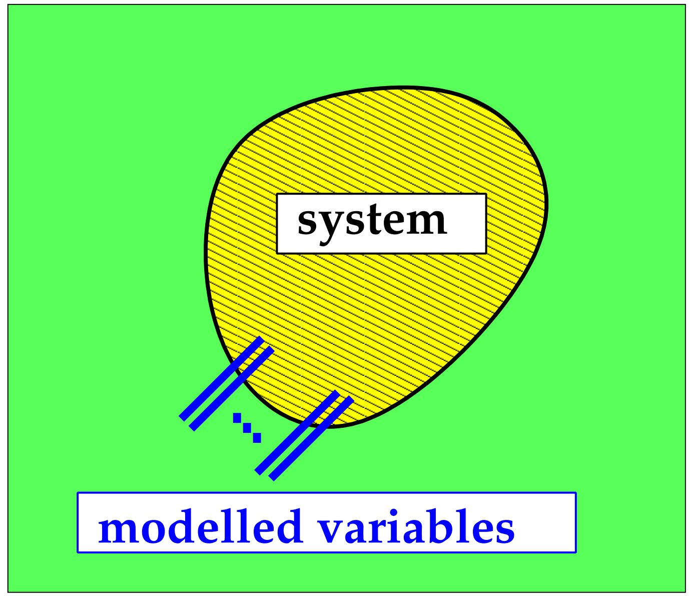
Deterministic case: $x(t+1) = f(x(t))$ **or** $\frac{d}{dt}x = f(x)$, $\quad w = h(x)$.

Stochastic case: $x(t+1) = f(x(t), \varepsilon(t))$,

or $dx = f(x)\,dt + h(x)\,d\varepsilon, \qquad w = h(x)$.

ε: **internal noise**

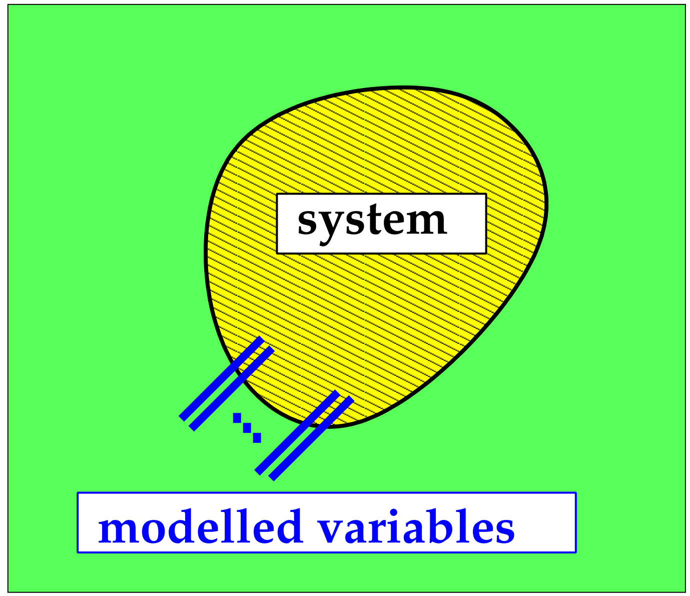
Closed systems



But closed systems do not form a good model class:

- **they do not cope with interconnection, with tearing**
- **the basic laws of physics are not closed systems**
- **implicitly forces to model the environment**

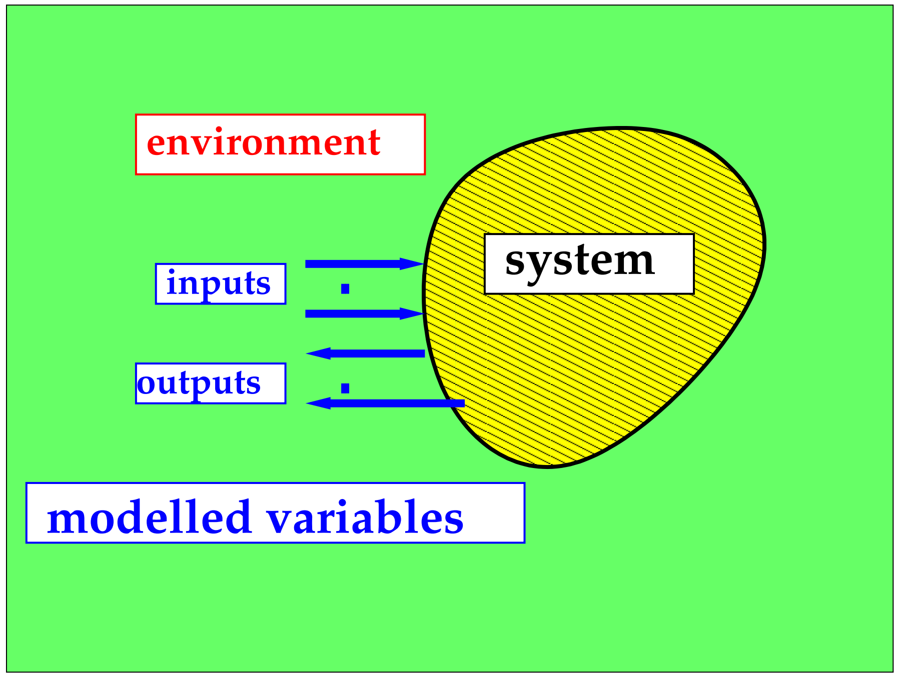
Closed systems



How to model interaction with the environment?

Open systems

Classical approach:



$\rightsquigarrow \quad x(t+1) = f(x(t), u(t)),\ y(t) = h(x(t), u(t)), w = (u, y),$ **or**
$x(t+1) = f(x(t), u(t), \varepsilon(t)),\ y(t) = h(x(t), u(t), \varepsilon(t)), w = (u, y)$**,**
or transfer functions, or ARMAX systems,...

Open systems

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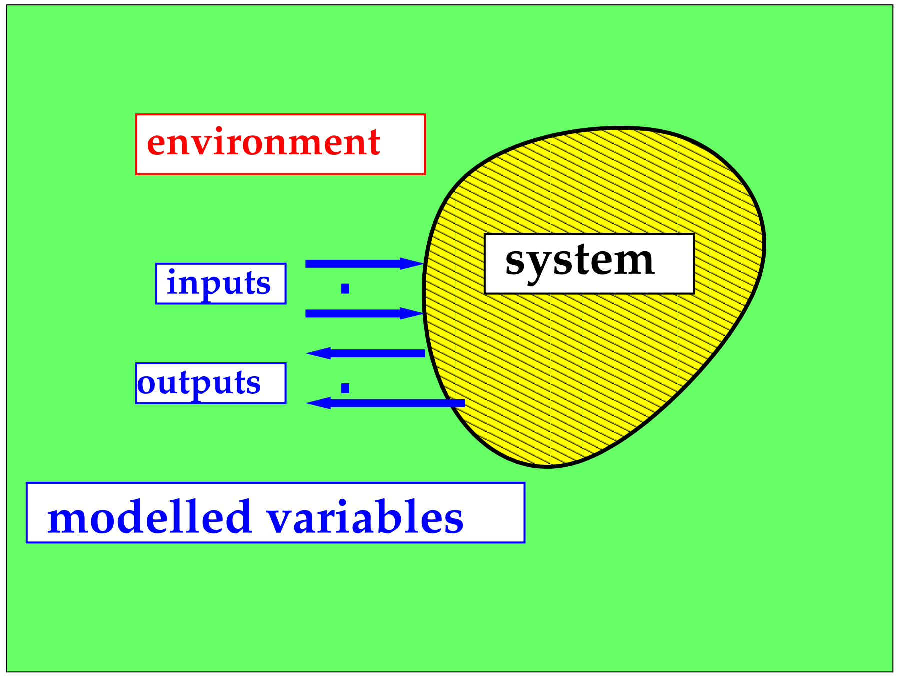

$\leadsto \quad x(t+1) = f(x(t), u(t)),\ y(t) = h(x(t), u(t)), w = (u, y),$ **or**
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Does this input/output partition respect the physics?

Input/output thinking

The input/output view as the primary and universal concept for open systems is a misconception
It fails in the first examples.

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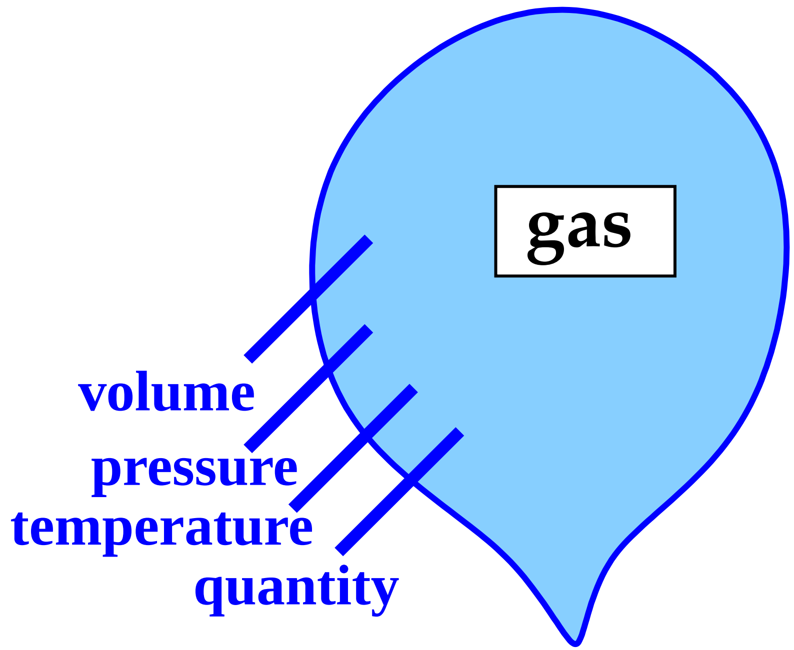
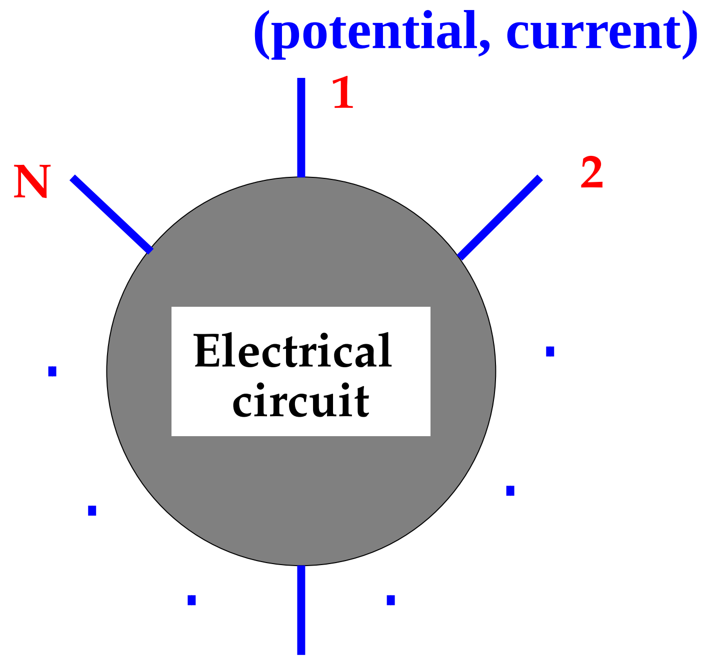


The gas law imposes the relation on $PV = NT$.

It makes no sense to view this in an input/output way.

Input/output thinking

The input/output view as the primary and universal concept for open systems is a misconception. It fails in the first examples.



The circuit imposes a relation on

$$V_1, I_1, V_2, I_2, \ldots, V_\mathtt{N}, I_\mathtt{N}$$

Only after modeling $\Rightarrow$ voltage or current driven terminals.

Input/output thinking

Maxwell's equations

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\ , \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\ , \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

10 variables, 8 equations, $\Rightarrow \exists$ free variables.

But it makes no sense to declare some variables as inputs...

Input/output thinking

The input/output view as the primary and universal concept for open systems is a misconception
It fails in the first examples.

The strongest argument against input/output thinking comes from system interconnection

variable sharing not output-to-input assignment

is the mechanism to interconnect systems.

BEHAVIORS

Behavioral systems - deterministic case

A (static) model is a subset $\mathscr{B}$ of the universum $\mathscr{U}$ of possible outcomes of a phenomenon.

$\mathscr{B}$ is the behavior of the model.

A dynamical system $:\Leftrightarrow$ $(\mathbb{T}, \mathbb{W}, \mathscr{B})$, **with**

$\mathbb{T} \subseteq \mathbb{R}$	**the time set**
$\mathbb{W}$	**the signal space**
$\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$	**the behavior**

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So, a dynamical system is merely a family of time-trajectories taking values in a signal space.

If $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$**, then all variables are treated on the same level. When analyzing** $\mathscr{B}$**, some components of** $w \in \mathscr{B}$ **may be 'free', in a sense 'inputs'.**

Behavioral systems - deterministic case

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A rich theory has been developed in this deterministic case, featuring new viewpoints, e.g. about LTIDSs, about controllability, etc.

Linear time-invariant difference systems

The dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **is said to be**

- **linear** $:\Leftrightarrow \mathscr{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ **is linear**
- **time-invariant** $:\Leftrightarrow \mathscr{B} = \sigma\mathscr{B}$
- **complete** $:\Leftrightarrow$

$$[\![w \in \mathscr{B}]\!] \Leftrightarrow [\![w|_{[t_1,t_2]} \in \mathscr{B}|_{[t_1,t_2]} \textbf{ for all } t_1, t_2 \in \mathbb{Z}]\!]$$

Linear time-invariant difference systems

The following are equivalent for $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$

- **Σ is linear, time-invariant, complete**
- $\mathscr{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ **linear, shift-invariant, and closed**
- $\exists$ **a polynomial matrix** $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **such that**

$$\mathscr{B} = \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} \mid R(\sigma)w = 0\}$$

that is, $\mathscr{B}$ **is the solution set of**

$$R_0 w(t) + R_1 w(t+1) + \cdots + R_{\mathtt{L}} w(t+\mathtt{L}) = 0 \text{ for all } t \in \mathbb{Z}$$

'kernel representation'

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- R' **and** R'' **define the same system**

  $\Leftrightarrow$ **their rows generate the same** $\mathbb{R}[\xi, \xi^{-1}]$**-module**

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- R' **and** R'' **define the same system** $\Leftrightarrow$ **their rows generate the same** $\mathbb{R}[\xi, \xi^{-1}]$**-module**

- $\exists$ **one-to-one relation between LTIDSs and** $\mathbb{R}[\xi, \xi^{-1}]$**-modules**

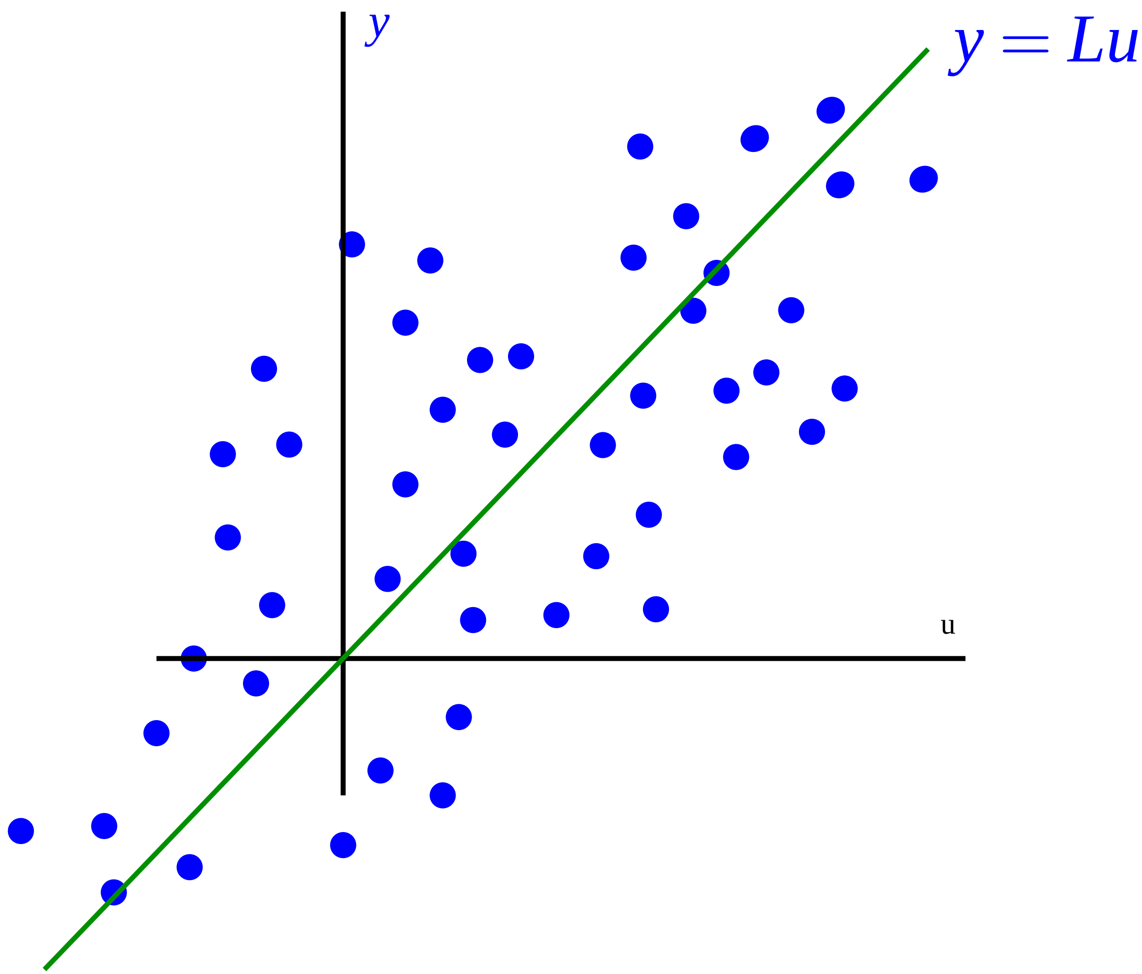
STOCHASTIC BEHAVIORS

STATIC CASE

Static case

'Regression'

$$y = Lu + \varepsilon$$



Static case

'Regression' $$y = Lu + \varepsilon$$

ε models the uncertainty of the 'law'

$$y = Lu$$

Classical: ε is a random vector.

But what should one assume about u?

And about the relation between u and ε?

Static case

'Regression' $y = Lu + \varepsilon$

Classical: ε is a random vector.

But what should one assume about u?

And about the relation between u and ε?

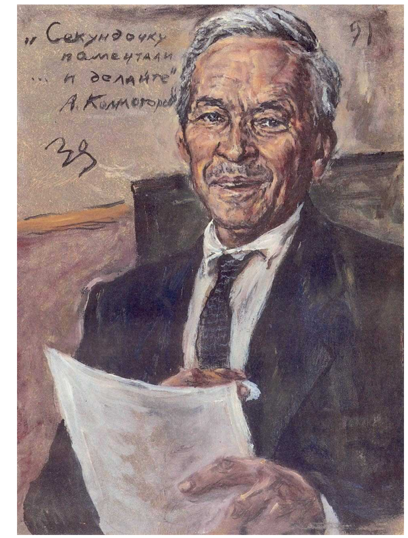
Since u is 'external', generated by the environment, one should not state anything about u.

Modeling a system should not require modeling the environment!

We also want to treat u and y on the same level

Stochastic static linear system

Recall the classical definition of an abstract random variable $(\mathbb{A}, \mathscr{A}, P)$ **with**



$\mathbb{A}$ **the space of elementary events**

$\mathscr{A}$ **a sigma-algebra of subsets of** $\mathbb{A}$

$P : \mathscr{A} \rightarrow [0, 1]$ **a probability measure**

In what is called an $\mathtt{n}$**-dimensional real random vector, we obtain** $(\mathbb{R}^{\mathtt{n}}, \mathscr{A}, P)$

with $\mathscr{A}$ **the sigma-algebra of Borel subsets of** $\mathbb{R}^{\mathtt{n}}$**.**

Our proposal is that (even for regression!), we should not take the Borel sigma-algebra.

Stochastic static linear system

Definition: A stochastic static linear system is a random variable

$$(\mathbb{R}^{\mathtt{n}}, \mathscr{A}, P)$$

with $\mathscr{A}$ the sigma-algebra of subsets of $\mathbb{R}^{\mathtt{n}}$ defined as follows in terms of a linear subspace $\mathbb{L} \subseteq \mathbb{R}^{\mathtt{n}}$

$$\mathscr{A} = \{\mathbb{S} \subseteq \mathbb{R}^{\mathtt{n}} \mid \mathbb{S} = \mathbb{S}' + \mathbb{L}, \mathbb{S}' \subseteq \mathbb{R}^{\mathtt{n}} \textbf{ Borel}\}$$

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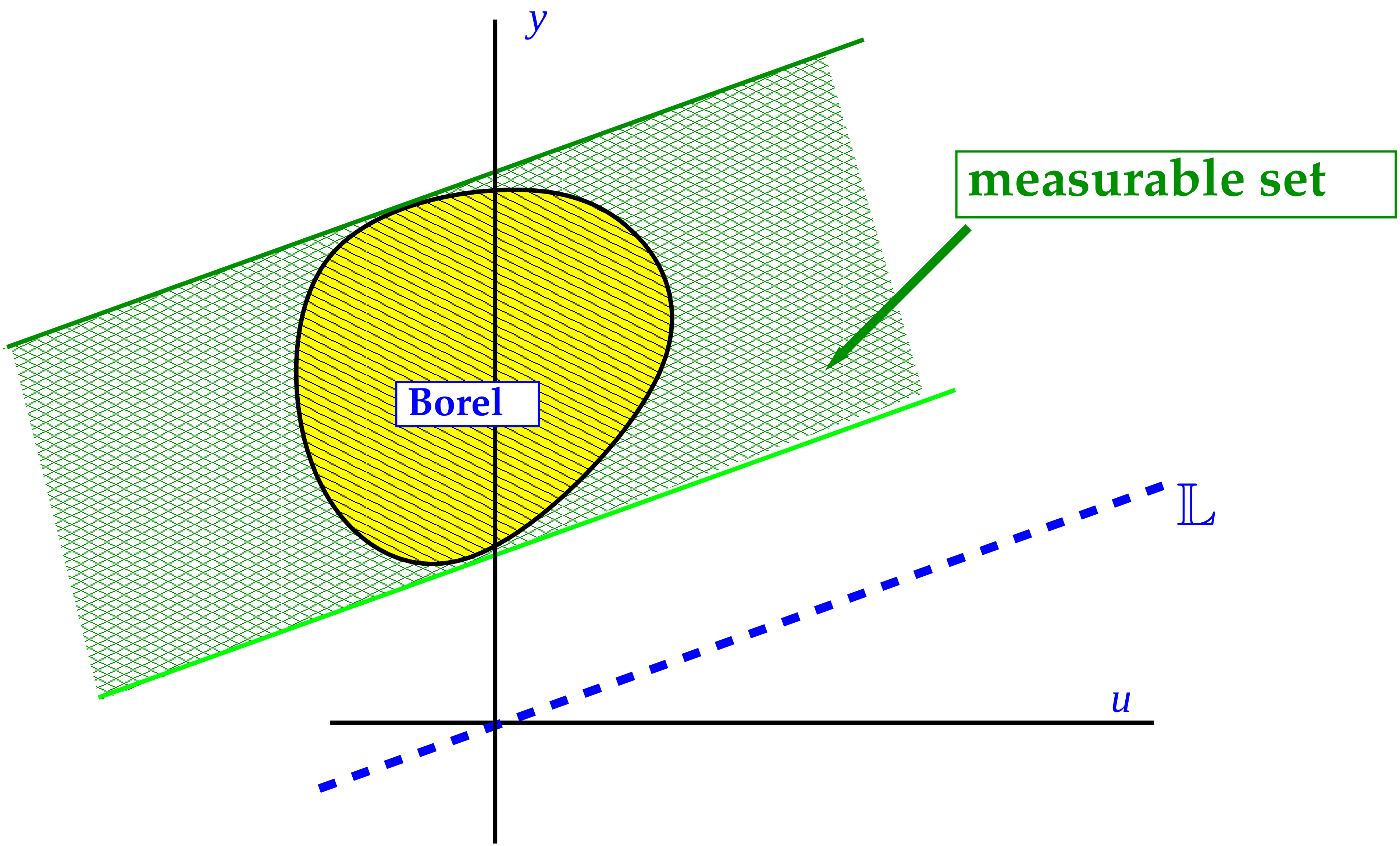
<u>Special cases</u>:

$\mathbb{L}$	$= \{0\}$	**classical random vector**
$P(\mathbb{L})$	$= 1$	**deterministic case**

In pictures

Sets for which the probability is defined:



Representation

A stochastic static linear system on $\mathbb{R}^{\mathtt{w}}$ admits a representation

$$Rw = \varepsilon$$

with R a real matrix and
ε a classical real random vector.

Special cases:

$R = I \quad \rightsquigarrow \quad w = \varepsilon$ **classical random vector**

$\varepsilon = 0 \quad \rightsquigarrow \quad Rw = 0$ **deterministic system**

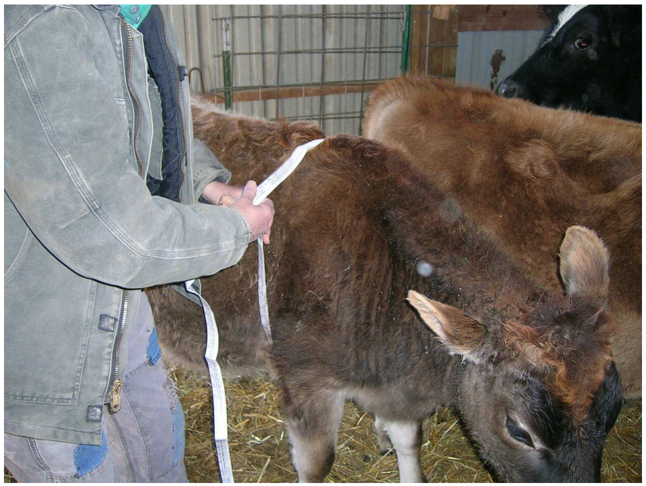
$\mathtt{dimension}(\mathtt{R}) =$ **degrees of freedom.**

Regression

- **Case $\mathtt{n} = 2$. Def. says that** $y - \alpha u$ **is random but that** u **and** y **are NOT random variables. (in the formal sense that the projections are not 'measurable' maps.)**
- **This is the intention of a regression model. There is no claim in such a model that** u **is random or deterministic, or that** ε **is dependent or independent of** u **or** y**.**

Examples

How do you weigh a cow?

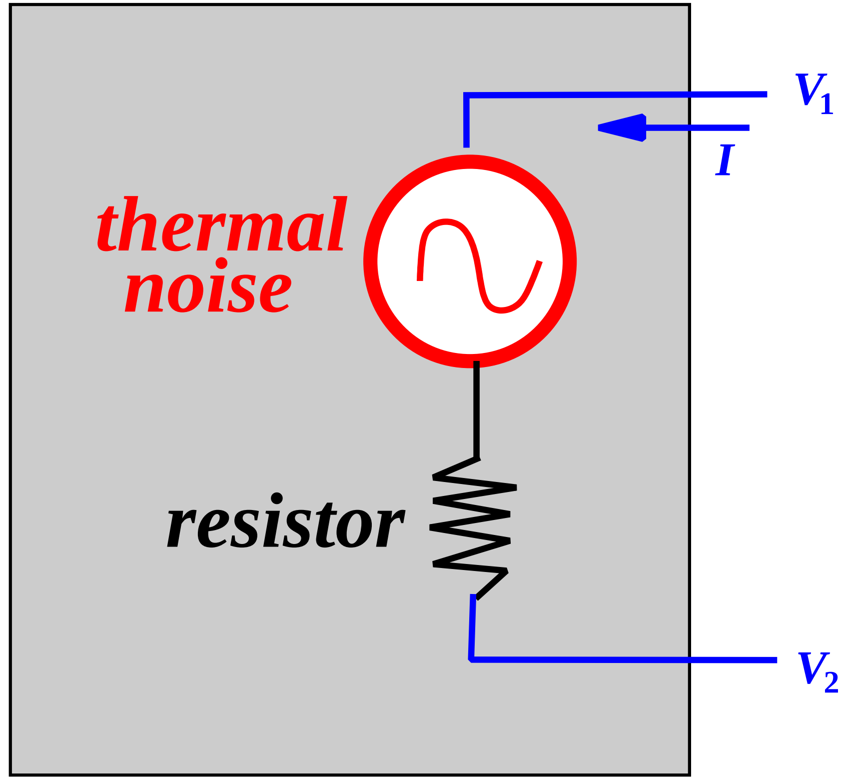


$$\textbf{weight} - \alpha\ \textbf{circumference}$$

is a random variable, not the weight or the circumference.

Examples

Johnson-Nyquist resistor noise

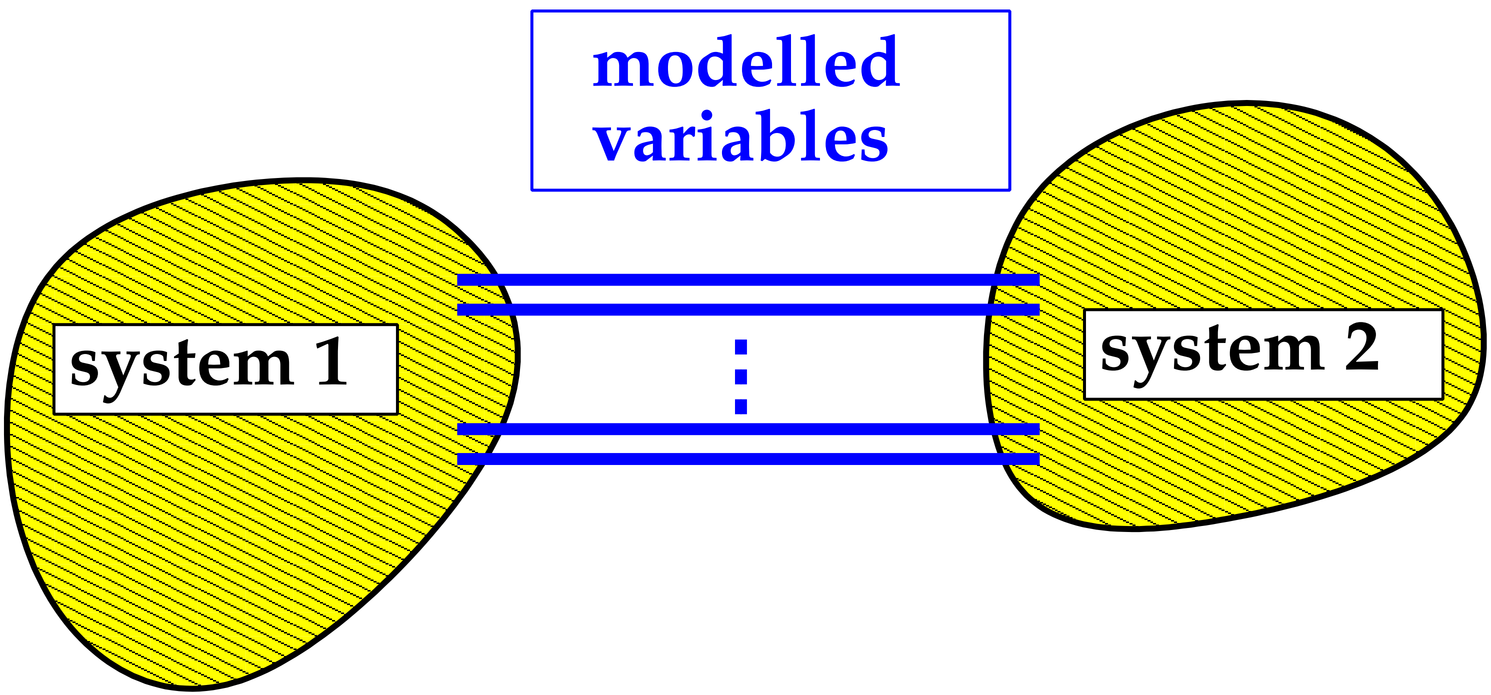


$$V_1 - V_2 - RI = V_{\text{noise}}$$

with V_{noise} **a random variable**

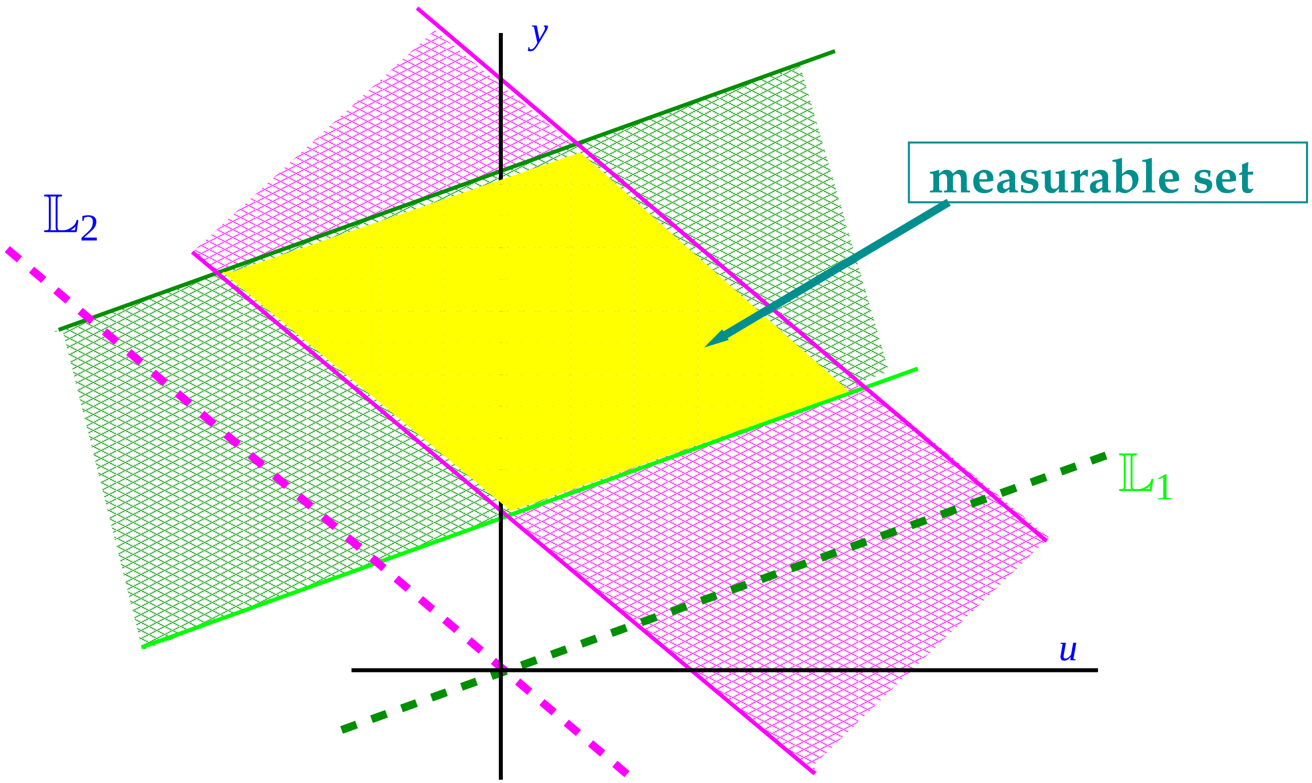
Interconnection

After interconnection, i.e., after modeling the environment, we obtain



Interconnection

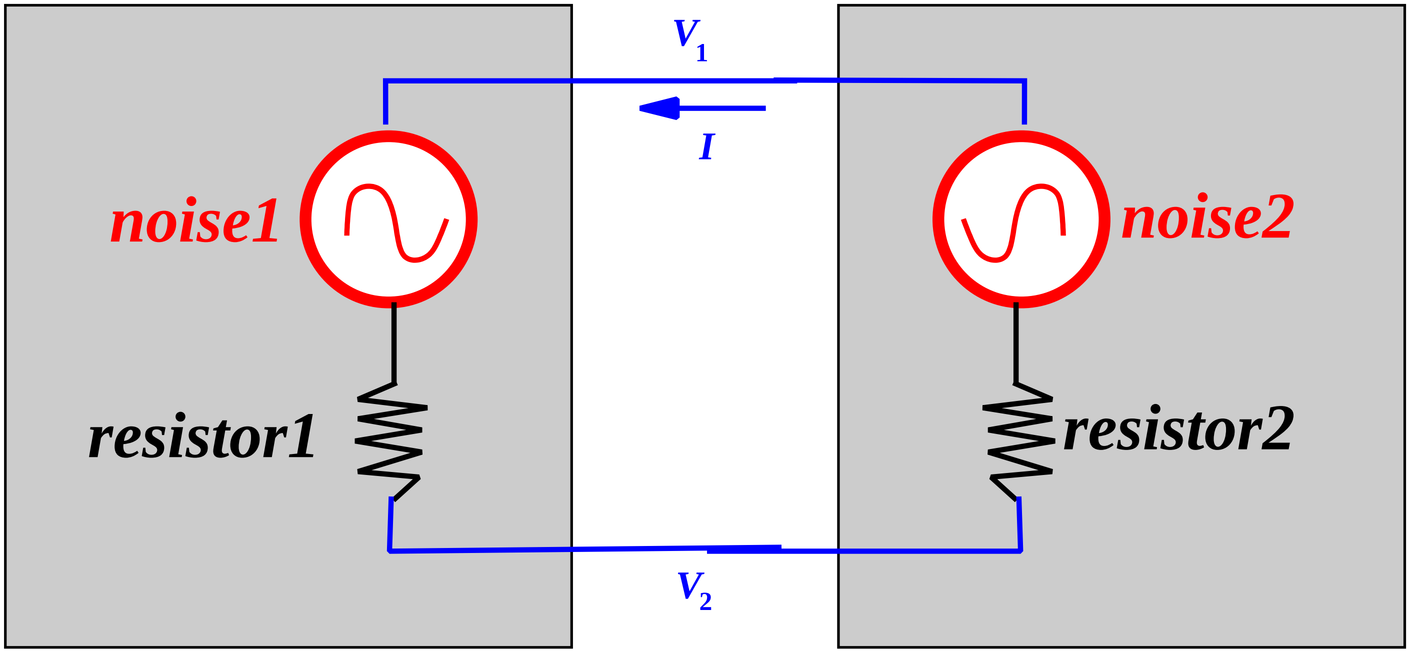
Leading to the σ-algebra generated by the intersections, and the product measure:



<u>Special case</u>: $\mathbb{L}_2 = \{u = 0\}$,

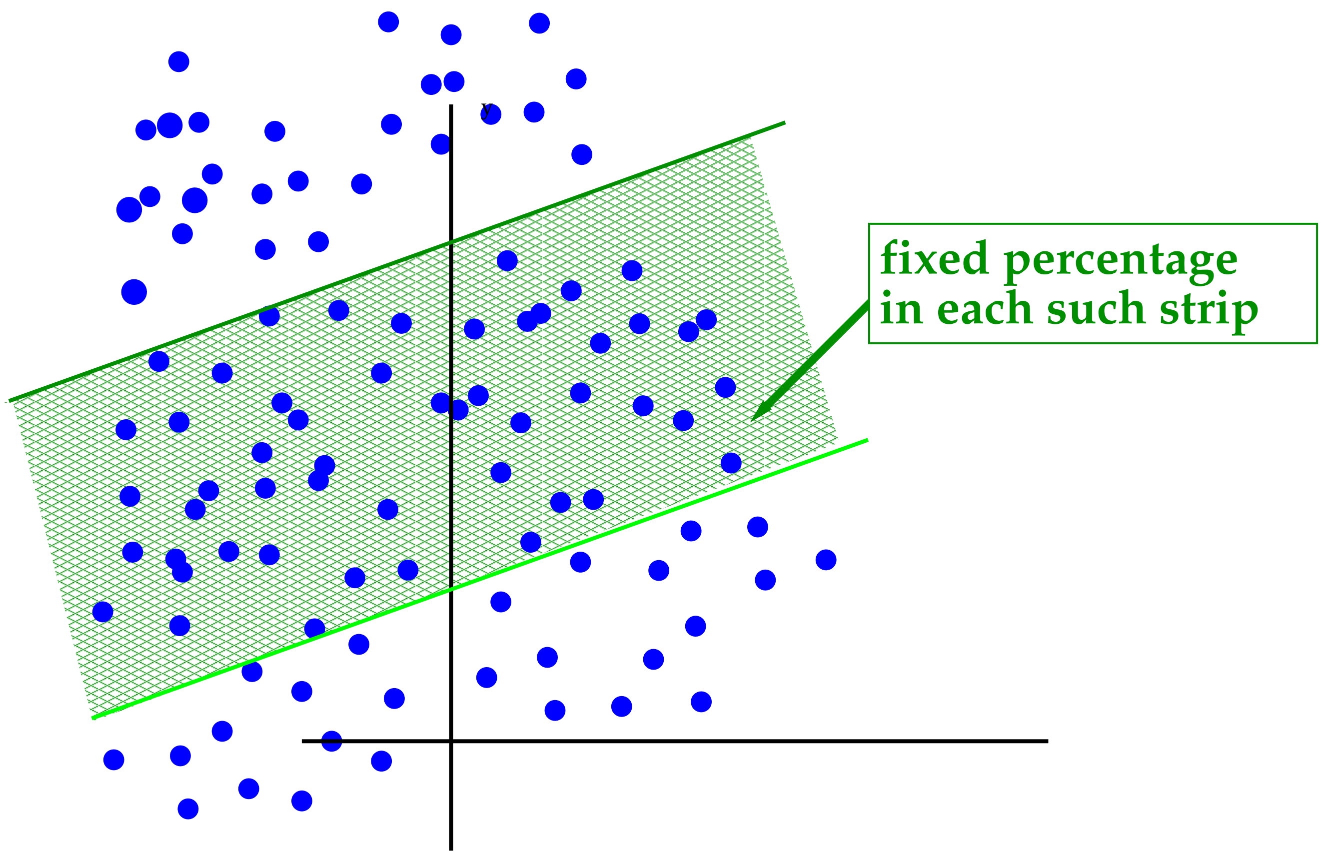
u **is then a random variable independent of** ε_1**.**

Example



$$\rightsquigarrow \qquad I = \frac{\varepsilon_1 + \varepsilon_2}{R_1 + R_2} \qquad V_1 - V_2 = \frac{(R_1+R_2)\varepsilon_1 + R_1\varepsilon_2}{R_1+R_2}$$

Regardless of the experimental conditions (i.e., of the interconnection)



STOCHASTIC BEHAVIORS

DYNAMIC CASE

Stochastic linear time-invariant system

A **stochastic linear time-invariant dynamical system** is given by a stationary random process ε and a polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$.

The behavior consists of all $w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}}$ such that

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In particular, there exists $\mathscr{M}$, an $\mathbb{R}[\xi, \xi^{-1}]$-submodule of $\mathbb{R}[\xi, \xi^{-1}]^{\mathtt{w}}$, such that

$$[\![f \in \mathscr{M}]\!] \ \Rightarrow \ [\![f^{\top}(\sigma, \sigma^{-1})\, w \text{ is a stationary process}]\!]$$

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In fact, $\mathscr{M}$ = the module generated by the transposes of the rows of R.

If $f^\top = hR$, then $f(\sigma,\sigma^{-1})\,w = h(\sigma,\sigma^{-1})\,\varepsilon$.

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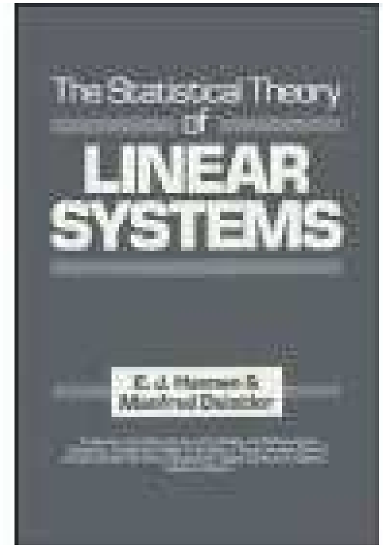
<u>To be worked out</u>:

Representation questions, their uniqueness, system identification issues, . . .

MODELING DISTURBANCES

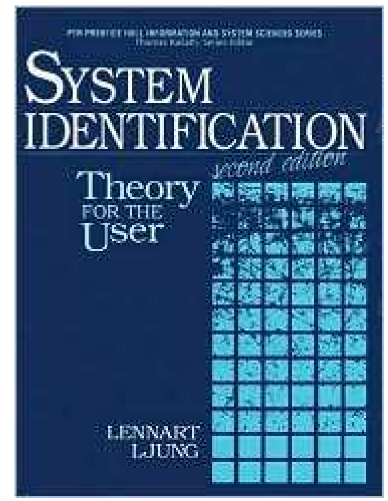
AS STOCHASTIC PROCESSES

Stochastics in ARMAX systems



$$A(\sigma)y = X(\sigma)u + M(\sigma)\varepsilon$$

The mathematics behind ARMAX systems are among the most elegant, appealing, and subtle in system theory.



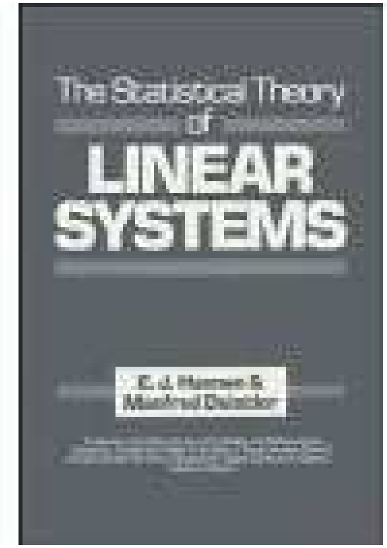
But what about the modeling aspect?

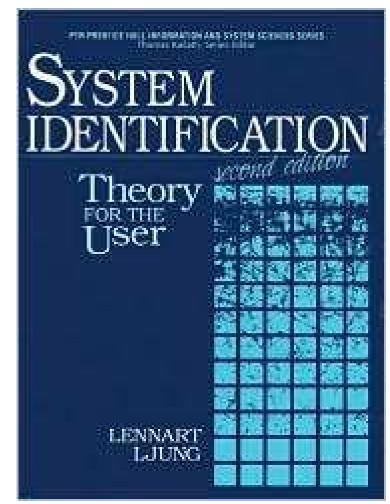
Stochastics in ARMAX systems

$$A(\sigma)y = X(\sigma)u + M(\sigma)\varepsilon$$

What is the rationale of assuming that the disturbances ε are stochastic processes?

Should one interpret probability in the sense of relative frequency? or in the sense of degree of belief?





Degree of belief

If probability in ARMAX system identification is to be interpreted in the sense of degree of belief, then

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- **what is the sense of worrying about consistency and asymptotic efficiency in SYSID?**
- **why should we care about their degree of belief?**
- **why not simply stick to least squares, and be much more parsimonious in expressing beliefs?**

Relative frequency

When there is a clear existing real ensemble, relative frequencies are clear and real. Is this the case in time-series and uncertain dynamical systems?

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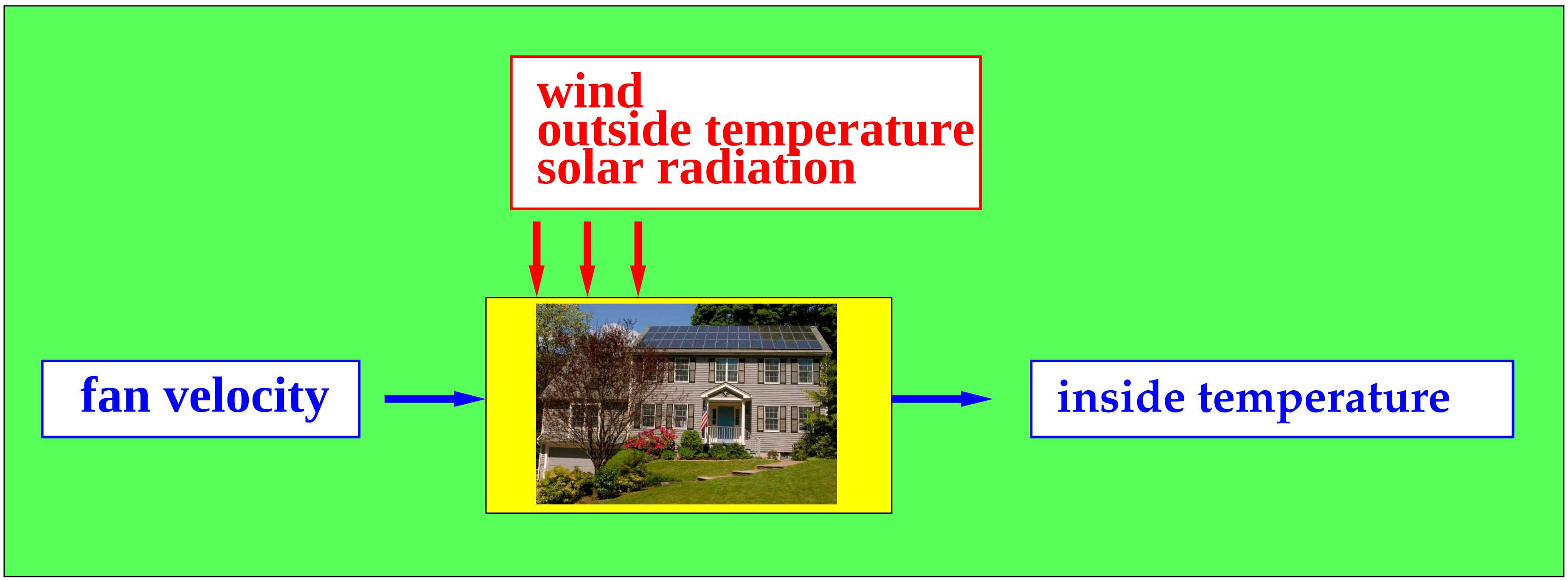
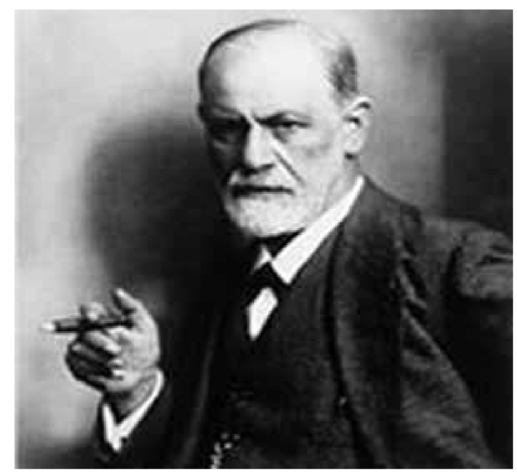


Are these 'disturbances' stochastic processes, even approximatey? If so, why?

Uncertainty

The universal use of probability as a panacea for modeling uncertainty in systems and control (and elsewhere) is for me a constant source of discomfort, for a feeling of Das Unbehagen in der Kultur



Is probabilty real?

What is the probability of heads ?

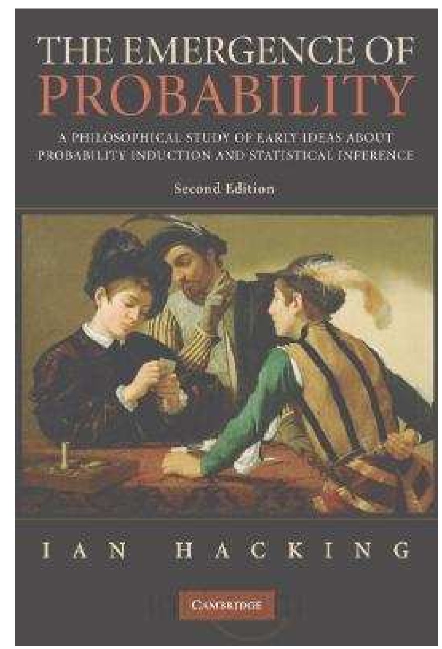
Many seem to believe that the randomness is

in the coin !

Is probabilty real?

What is the probability of heads ?

"The propensity to give heads is as much a property of the coin as its mass, and the stable long run frequency found on repeated trials is an objective fact of nature independent of anyone's knowledge of it" **I. Hacking, p. 14.**



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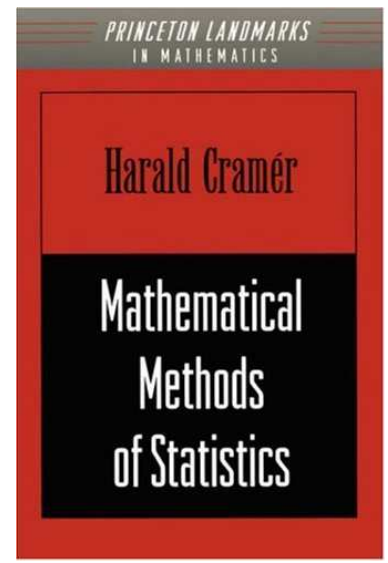
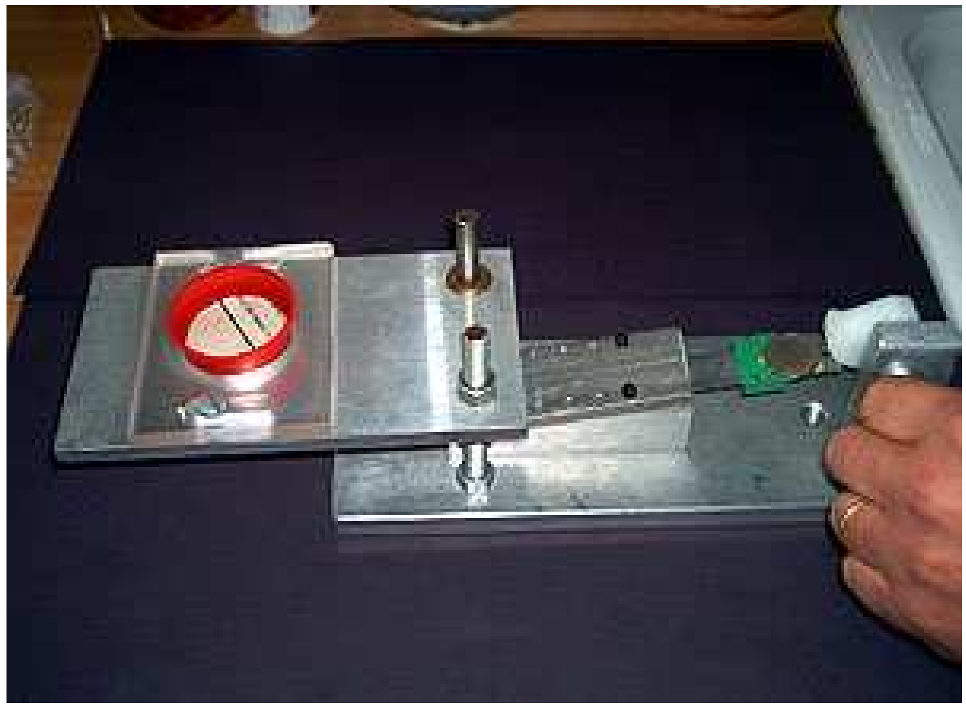


"The numbers $p_{\mathtt{r}}$ **[the probability of the outcome $\mathtt{r}$]** *should in fact be regarded as physical constants of the particular die that we are using"* **H. Cramer, p. 154**

Physics or stochastics?

Persi Diaconis builds a coin tosser



and discovered that if the coin is tossed exactly the same way, it falls on the same side 100% of the time.

Physics or stochastics?

The press appears indignified:

The Not So Random Coin Toss

Listen by David Kestenbaum

Larger Image of the Machine

Susan Holmes

Statistician Persi Diaconis' mechanical coin flipper.

All Things Considered, February 24, 2004 ·
Flipping a coin may not be the fairest way to settle disputes. About a decade ago, statistician Persi Diaconis started to wonder if the outcome of a coin flip really is just a matter of chance. He had Harvard University engineers build him a mechanical coin flipper. Diaconis, now at Stanford University, found that if a coin is launched exactly the same way, it lands exactly the same way.

The randomness in a coin toss, it appears, is introduced by sloppy humans. Each human-generated flip has a different height and speed, and is caught at a different angle, giving different outcomes.

Physics or stochastics?

The scientists come to the following conclusions:

We conclude that coin tossing is ‘physics’, not ‘random’

P. Diaconis, S. Holmes and R. Montgomery, Dynamical bias in the coin toss,

SIAM Review, **2007, page 211.**

I could have told them that without the benefit of a machine...

Physics or stochastics?

The scientists come to the following conclusions:

If we have this much trouble analyzing a common coin toss, the reader can imagine the difficulty we have with interpreting typical stochastic assumptions in econometric analysis

Agreed, from the bottom of my heart!

CONCLUSIONS

Conclusions

- **An open stochastic system is best defined in terms of unusual σ-algebra.**
  $\rightsquigarrow$ a crisper definition, which does not require input/output separation,
  and avoids the discussion of statistical dependence of input and noise.

Conclusions

- **I am uncomfortable with the use of probability as a panacea for uncertainty.**

Conclusions

- **I am uncomfortable with the use of probability as a panacea for uncertainty.**
- **I find it difficult to fathom the origin of the conviction that uncertainty is intrinsic in some systems, e.g., coins and dice, and wiggly time-series. Comes (in part) from misunderstanding 'closed' versus 'open' systems.**

Possible exception: QM.

Copies of the lecture frames will be available from/at

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Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Manfred, enjoy your retirement!