Time Series Models With Asymmetric Laplace Innovations

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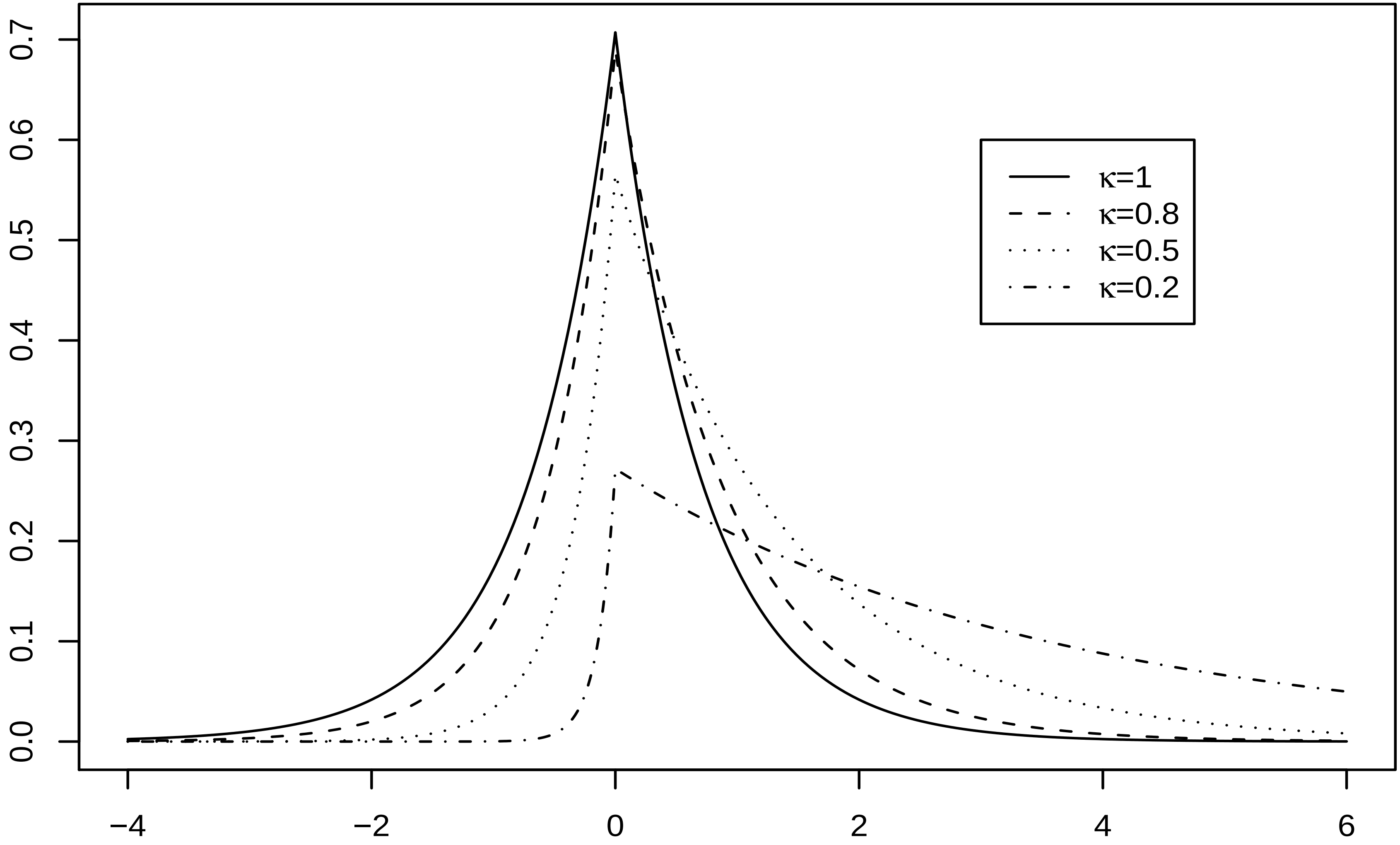
Summary

- Propose ARMA and GARCH models driven by AL noise.
- Suited for data that is **skewed**, **peaked**, and **leptokurtic** (kurtosis>3), but have higher order moments.
- For ARMAs: derive marginal and joint distributions; construct exact BLP confidence bounds.
- ARMA & GARCH estimation via conditional MLE.
- Evidence from case study of real estate returns suggests:
  - provide a competitive fit to models with (skewed) normal and t noise;
  - lower AIC and less parameters than normal noise models

The AL Distribution (Kotz *et al.*, 2001)

- AL plays in **geometric-stable** class, analogous role played by normal in **alpha-stable** class.
  - alpha-stable: **Normal** is only member with finite variance ($\alpha = 2$).
  - geom-stable: **AL** is only member with finite variance ($\alpha = 2$).
- Introduce skewness as in Fernandez and Steel (1998); applies to any symmetric dist e.g. normal, student-t.
- The pdf $f(y)$ of $Y \sim \mathcal{AL}(\theta, \kappa, \tau)$, has parameters: **location** θ, **scale** $\tau > 0$, **skewness** $\kappa > 0$.

$\mathcal{AL}(\theta = 0, \kappa, \tau = 1)$ Density Functions



Literature Review: AL in ARMA models

$$\phi(B)X_t = \lambda(B)Z_t$$

- Dewald & Lewis (1985), Novković (1998):
  - $X_t \sim$ Laplace $\Longrightarrow Z_t \sim$ mixture of Laplace pdf's.
- Jayakumar & Kuttykrishnan (2007):
  - $X_t \sim$ AL $\Longrightarrow Z_t \sim$ mixture of AL pdf's.
- Damsleth & El-Shaarawi (1989):
  - $Z_t \sim$ Laplace $\Longrightarrow X_t \sim$ mixture of Laplace pdf's.
- **Trindade, Zhu, Andrews (2009):**
  - $Z_t \sim$ AL $\Longrightarrow X_t \sim$ NOT mixture of AL pdf's.

Def: ARMA Model Driven by AL Noise

With $\theta = \theta(\kappa, \tau)$ s.t. $\mathbb{E}Z_t = 0$, $\{X_t\}$ is ARMA(p, q) if:

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-1} + \sum_{j=1}^{q} \lambda_j Z_{t-j} + Z_t, \quad \{Z_t\} \sim \text{IID } \mathcal{AL}(\theta, \kappa, \tau)$$

Assume $\{X_t\}$ is *nice*:

- **invertible** (ensures identifiability), $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$
- **causal** (independent of future):

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

Marginal Dist of ARMA With AL Noise

- For $E_1, E_2 \sim$ IID Exp(1), can express $Z_t \sim \mathcal{AL}(\theta, \kappa, \tau)$ as:

$$Z_t \stackrel{d}{=} \theta + \tau(E_1/\kappa - \kappa E_2)/\sqrt{2}$$

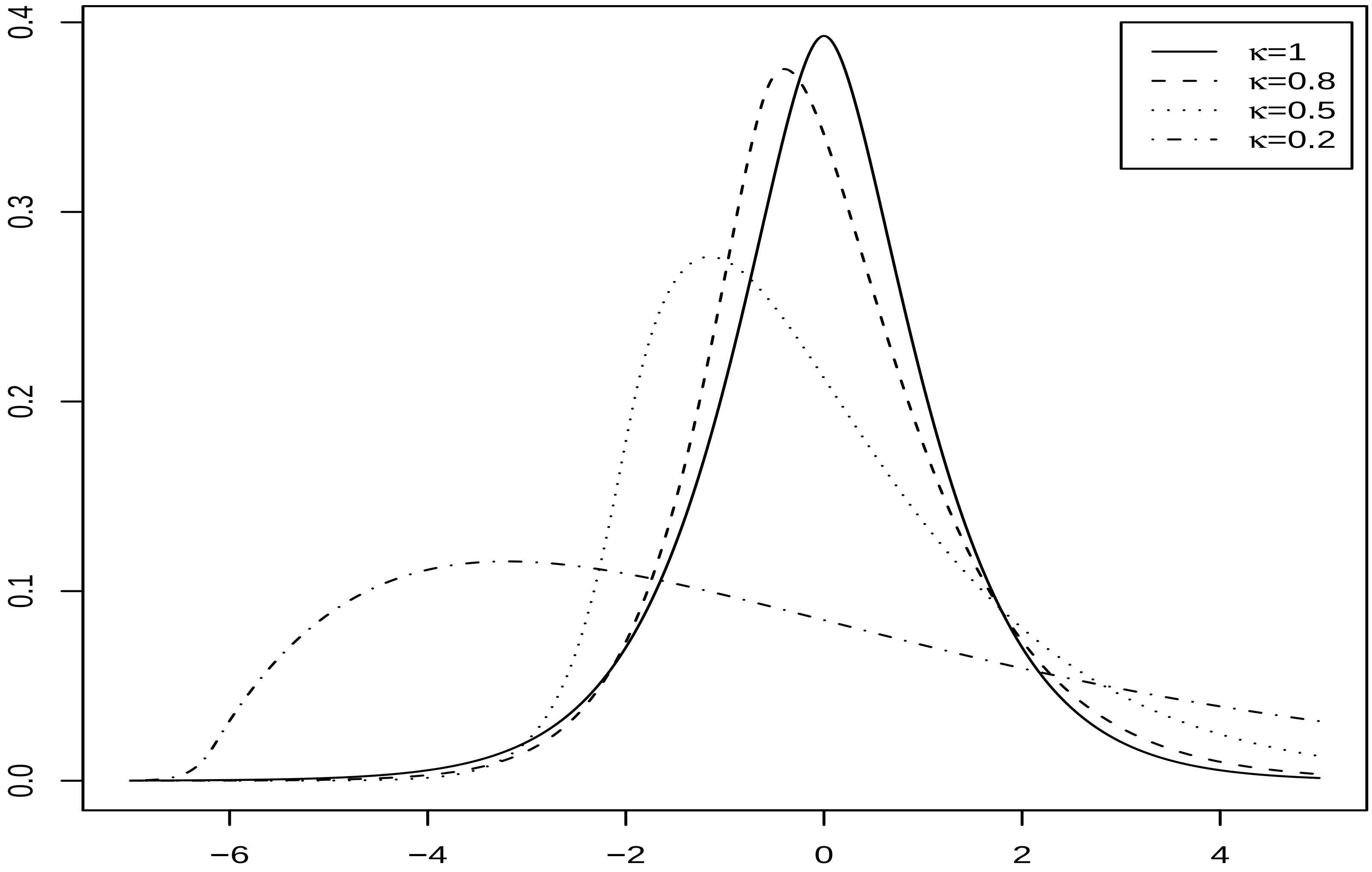
- Derive CDF of X_t using result about dist of linear combo of IID exponentials (Ali & Obaidullah, 1982)
Prop 1:

$$F_{X_t}(x) = \begin{cases} \sum_{i \in J_-} b_i e^{-(x-\nu)/a_i}, & x < \nu, \\ 1 - \sum_{i \in J_+} b_i e^{-(x-\nu)/a_i}, & x \geq \nu, \end{cases}$$

where ν, a_i, b_i depend on κ, τ, and the ψ_j's.

Ex: Marginal PDF of MA(1) With AL Noise

$\lambda = 0.8, \ \tau = 1$



Joint Dist of ARMA With AL Noise

Joint PDF of (X_t, X_{t-h}) can be obtained on case by case basis.

Ex: **ARMA(1,1).** With $\{W_1, W_2, W_3\}$ indep functions of Z_t's, can write

$$\begin{aligned} X_t &= \phi^h X_{t-h} + \lambda\phi^{h-1} Z_{t-h} + \sum_{j=0}^{h-1} \psi_j Z_{t-j} \\ &= W_1 + aW_2 + bW_3, \qquad a = \phi^h + \lambda\phi^{h-1},\ b = \phi^h \end{aligned}$$

- joint PDF of (W_1, W_2, W_3) is straightforward to derive from Prop 1, since each W_i is a linear combo of AL's;

Biv PDF of ARMA(1,1) With AL Noise

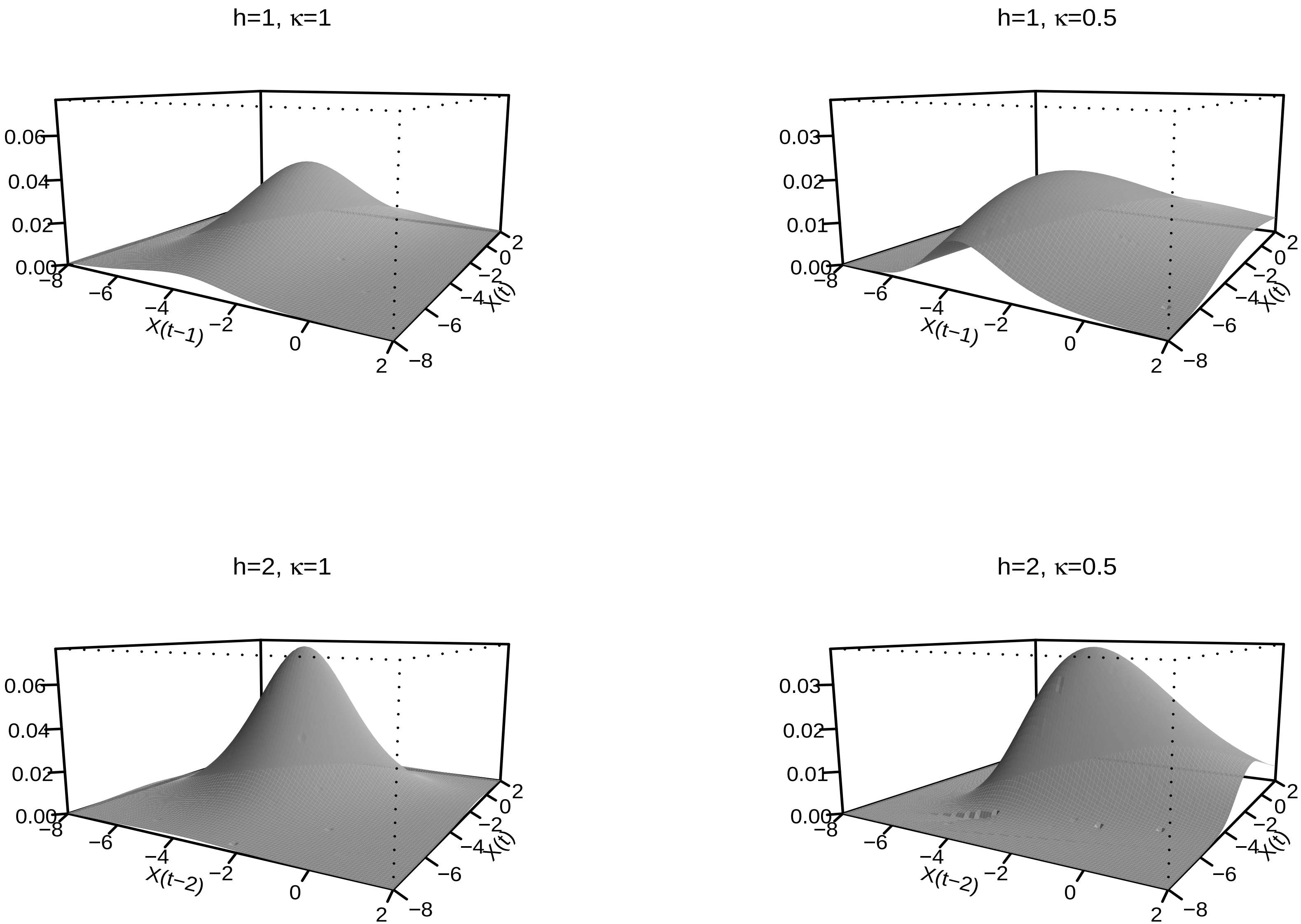
- have 1-1 transform with Jacobian $(b-a)^{-1}$:

$$(W_1, W_2, W_3) \mapsto (X_{t-h}, X_t, W_1)$$

- then obtain joint PDF of (X_{t-h}, X_t) by (numerically) integrating PDF of (X_{t-h}, X_t, W_1) over W_1, giving:

$$f_{X_{t-h},X_t,}(x,y) =$$
$$\frac{1}{|b-a|}\int_{-\infty}^{\infty} f_{W_1}(w) f_{W_2}\left(\frac{bx-y+w}{b-a}\right) f_{W_3}\left(\frac{y-w-ax}{b-a}\right) dw$$

Biv PDF of ARMA(1,1) With AL Noise



ARMA(p, q) Parameter Estimation

Use **conditional maximum likelihood**:

- log-likelihood of $\{X_{p+1}, \ldots, X_n | X_1, \ldots, X_p\}$:

$$l(\phi_1, \ldots, \phi_p, \lambda_1, \ldots, \lambda_q, \kappa, \tau) = \sum_{t=p+1}^{n} \log f(z_t; \kappa, \tau)$$

- z_t's are model residuals
- Resulting estimates commonly called *quasi-MLEs* **(QMLEs)**

ARMA QMLE Asymptotics

Consistency and asymptotic normality for general noise PDF $f(\cdot)$ derived under fairly mild assumptions by Li & McLeod (1988):

- requires (a.e.) existence and continuity of 1st and 2nd derivatives of $f(z;\kappa,\tau)$ with respect to both z and (κ,τ);
- not a problem since AL PDF is non-differentiable only at $z=\theta$;
- zero-mean restriction ($\mathbb{E}Z_t = 0$) needs care when computing 2×2 block of Fisher Info matrix corresponding to (κ,τ).

ARMA Best Linear Prediction (BLP)

BLP of X_{n+h} on infinite past is linear combo with min MSE:

$$\begin{aligned}\tilde{X}_{n+h} &= a_n X_n + a_{n-1} X_{n-1} + \cdots \\ &\equiv \sum_{j=h}^{\infty} \psi_j Z_{n+h-j}\end{aligned}$$

- coincides with *best* predictor on infinite past $\mathbb{E}(X_{n+h} | X_n, X_{n-1}, \ldots)$;
- expressible as linear combination of noise Z_t;
- quantiles of its distribution, obtainable from Prop 1, allow for construction of **exact prediction bounds**.

ARMA-GARCH Model With AL Noise

$\{X_t\}$ is the ARMA(p,q)

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-1} + \sum_{j=1}^{q} \lambda_j Z_{t-j} + Z_t$$

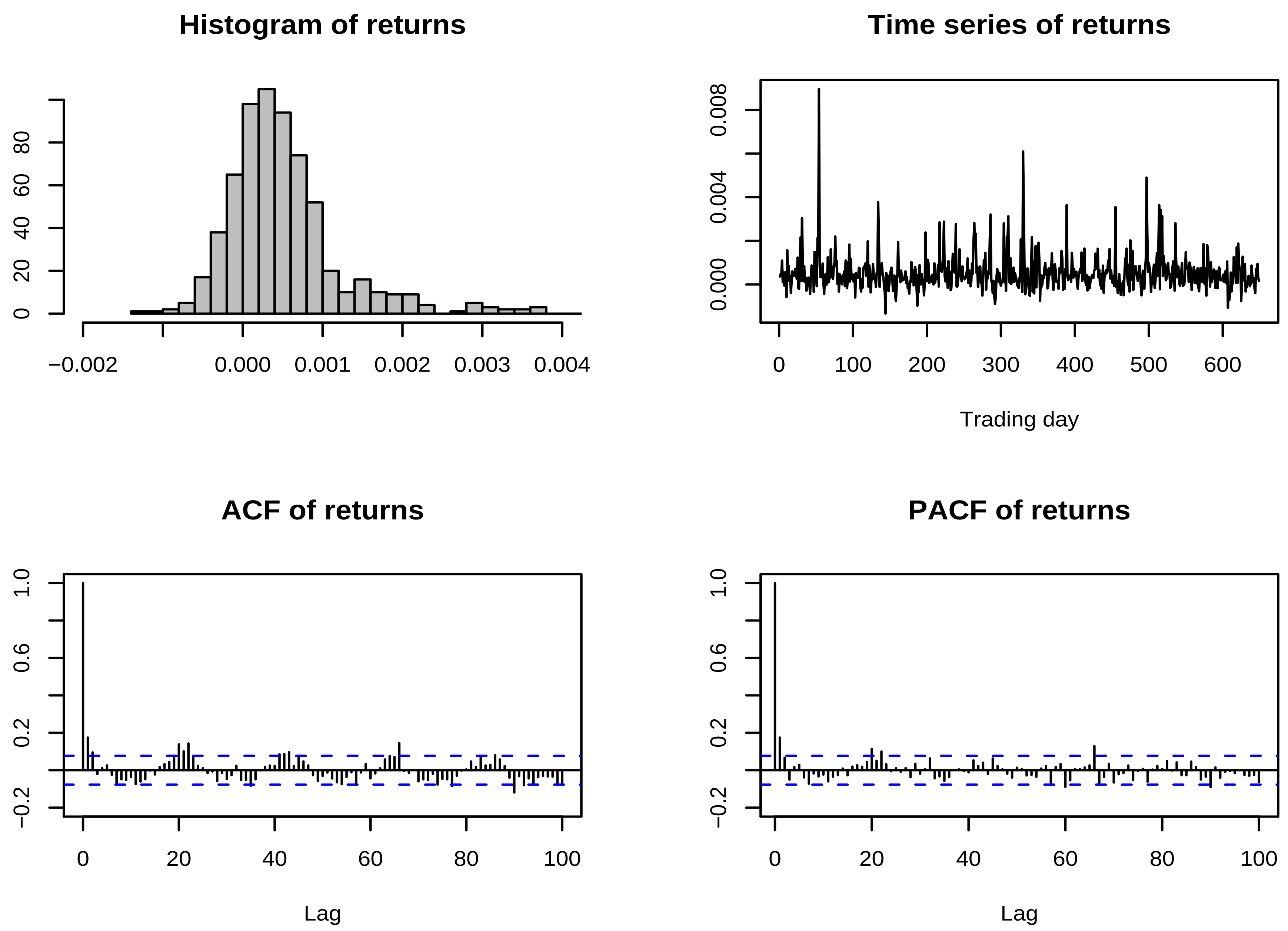
with $Z_t \sim$ GARCH(u,v) AL noise, defined by:

- $Z_t = \sigma_t e_t, \quad \{e_t\} \sim$ IID $\mathcal{AL}(\theta, \kappa, \tau)$;
- $\sigma_t^2 = \alpha_0 + \sum_{i=1}^{u} \alpha_i Z_{t-i}^2 + \sum_{j=1}^{v} \beta_j \sigma_{t-j}^2$;
- $\alpha_0 > 0$, $\alpha_i \geq 0$, $i = 1, \ldots, u$, and $\beta_j \geq 0$, $j = 1, \ldots, v$;
- $\theta = \theta(\kappa),\ \tau = \tau(\kappa)$ chosen so that $\{e_t\}$ has zero-mean, unit variance.

Case Study: TIAA-CREF Real Estate Fund

- Data Y_t: 15 June 2004 – 31 Dec 2006 (650 daily values).
- Returns X_t : $\quad X_t = \log Y_t - \log Y_{t-1}$.
- Peaked, heavy-tailed, right-skewed.
- ACF suggests weak monthly seasonal effect.
- Fit ARMA's; search for lowest AIC up to ARMA(15,15) with:
  - $Z_t \sim$ IID Normal, $\quad Z_t \sim$ IID AL;
  - $Z_t \sim$ GARCH(1,1) with innovations e_t:
    - Normal, skewed Normal, AL, skewed t.

TIAA-CREF Returns



Fitted Model Comparisons

Cond. Mean	Conditional Variance	AIC
ARMA(5,4)	IID Normal	-7391.3
ARMA(5,4)	Garch(1,1) Normal	-7465.1
MA(1)	Garch(1,1) Asym. Normal (AN)	-7578.4
MA(1)	**IID Asymmetric Laplace (AL)**	**-7703.7**
MA(1)	Garch(1,1) Asymm. Laplace (AL)	-7704.2
MA(1)	Garch(1,1) Asymm. Stud. t (AT)	-7724.2

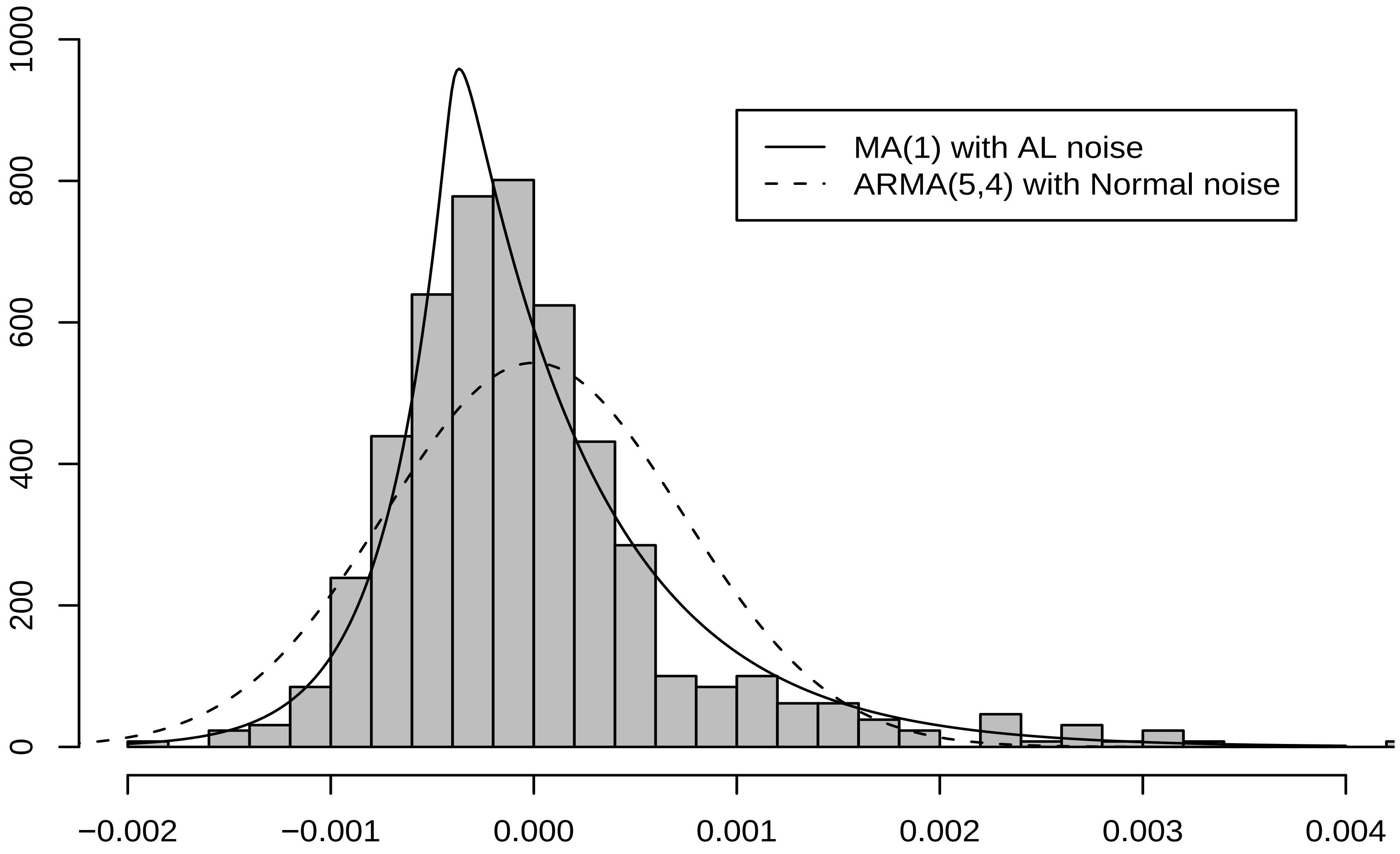
MA(1) & MA(1)-GARCH(1,1) Models

Noise Structure	Parameter Estimates (Standard Errors)					
	$\boldsymbol{\lambda}$	α_0	α_1	β_1	$\boldsymbol{\kappa}$	τ or ν
IID AL†	**0.062**				**0.66**	τ=6.31E-04
	(2.38E-02)				(3.68E-02)	(4.28E-05)
Garch AN*	**0.093**	2.97E-07	2.63E-01	2.24E-01	**1.80**	
	(4.65E-02)	(4.62E-08)	(8.71E-02)	(1.02E-01)	(1.03E-01)	
Garch AL*	**0.073**	4.97E-07	8.53E-02	1.00E-08	**1.50**	
	(1.51E-02)	(4.96E-08)	(4.22E-02)	(9.19E-02)	(2.01E-02)	
Garch AT*	**0.089**	6.59E-07	1.04E-01	1.00E-08	**1.52**	ν=3.06
	(2.97E-02)	(1.19E-07)	(6.21E-02)	(NaN)	(8.62E-02)	(0.42)

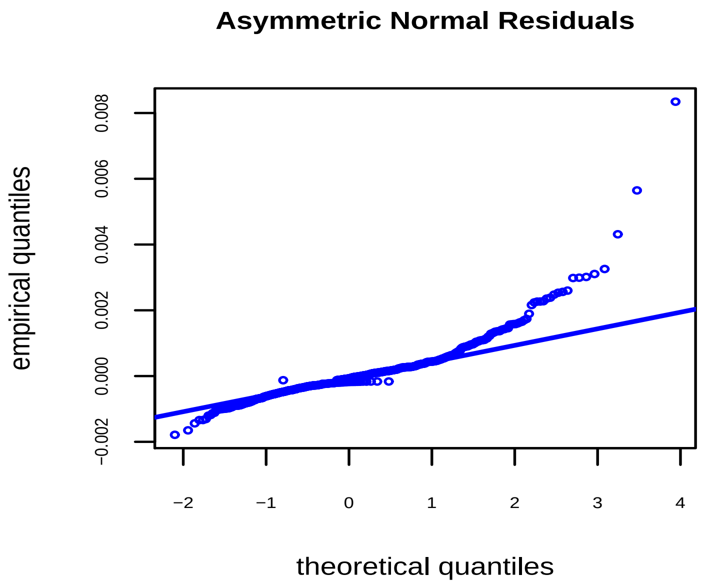
† Obtained via our own Matlab code; standard errors based on Fisher Information matrix.

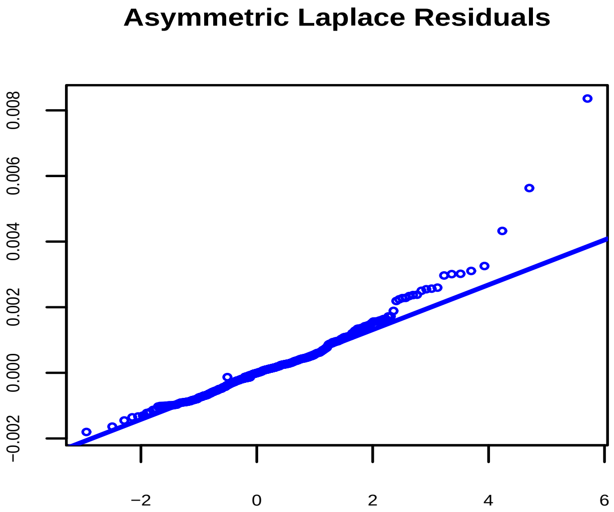
* Obtained via R package “fGarch” (Rmetrics); standard errors based on numerical Hessian.

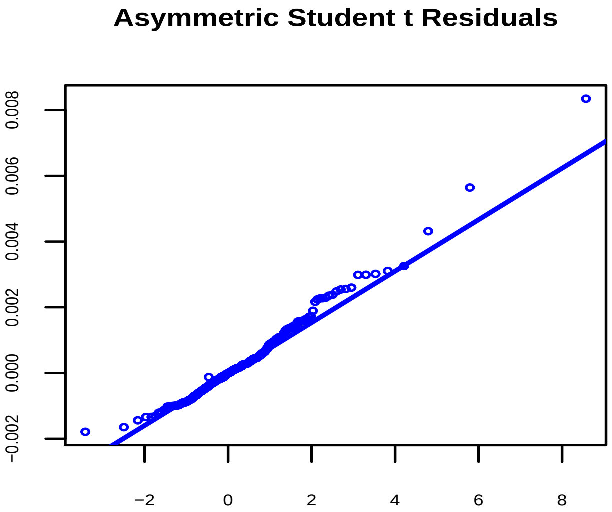
TIAA-CREF Marginal ARMA Model PDFs



MA(1)-GARCH(1,1) Residual QQ-Plots







Key References

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