On the Distribution of the Adaptive LASSO Estimator

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Penalized LS (ML) Estimators

Linear regression model $y = X\theta + \varepsilon$, n observations, k regressors.

$$\hat{\theta} = \arg\min_{\theta \in \mathbb{R}^k} \underbrace{\|y - X\theta\|^2}_{\text{likelihood or LS -part}} + \underbrace{\lambda_n\, p(\theta)}_{\text{penalty}}$$

λ_n is a tuning parameter.

- Bridge estimators (l_q - type penalties, Frank and Friedman, 1993, LASSO for $q = 1$, Tibshirani, 1996).
- Hard- and soft-thresholding estimators.
- Smoothly clipped absolute deviation (SCAD) estimator (Fan and Li, 2001).
- Adaptive LASSO estimator (Zou, 2006).

These estimators ($q \leq 1$ for Bridge est.) can be viewed to simultaneously perform model selection and parameter estimation.

Some terminology

- Conservative model selection – Zero coefficients are found with asymptotic probability less than 1.
- Consistent model selection – Zero coefficients are found with asymptotic probability equal to 1. An estimator performing consistent model selection is said to have the sparsity property.
- Oracle property – Asymptotic distribution coincides with the one of the infeasible unpenalized estimator using the true zero restrictions.

Consistent vs. conservative model selection is in our context driven by the asymptotic behavior of tuning parameters λ_n. ("Sparsely" vs. "non-sparsely" tuned procedures).

Literature on distributional properties of PLSEs

- Knight & Fu, 2000. Moving-parameter asymptotics for non-sparsely tuned LASSO and Bridge estimators in general.
- Fan & Li, 2001. Fixed-parameter asymptotics for SCAD.
- Zou, 2006. Fixed-parameter asymptotics for sparsely-tuned LASSO and adaptive LASSO.
- Additional papers establishing the oracle property for sparsely-tuned PLSEs and related estimators within a fixed-parameter framework.

  Fan & Li (2002, 2004), Bunea (2004), Bunea & McKeague (2005), Wang & Leng (2007), Li & Liang (2007), Wang, G. Li, & Tsai (2007), Zhang & Li (2007), Wang, R. Li, & Tsai (2007), Zou & Yuan (2008), Zou & Li (2008), Johnson, Lin, & Zeng (2008), ...

Literature on distributional properties of PLSEs (cont'd)

This talk is based on

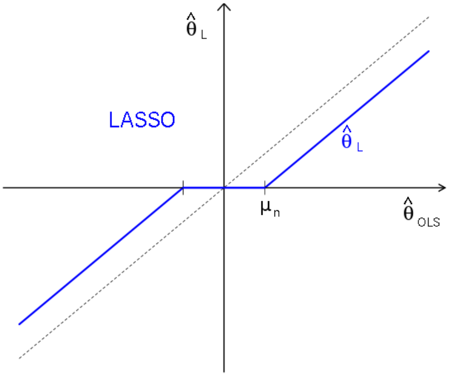
- Pötscher & Leeb, 2009. Finite-sample distribution, moving-parameter asymptotics for hard-thresholding, LASSO, and SCAD. Impossibility result for the estimation of the cdf.
- Pötscher & Schneider, 2009. Analogous results for the adaptive LASSO.
- Pötscher & Schneider, 2008. Finite-sample and asymptotic coverage probabilities of confidence sets for hard-thresholing, LASSO, ad. LASSO.

Definition of the (adaptive) LASSO estimator $\hat{\theta}_{AL}$

LASSO estimator (Tibshirani, 1996)

$$\hat{\boldsymbol{\theta}}_{L} = \underset{\boldsymbol{\theta}\in\mathbb{R}^k}{\arg\min} \; \|\mathbf{y} - X\boldsymbol{\theta}\|^2 + 2n\mu_n \sum_{i=1}^{k} |\theta_i| \qquad \mu_n > 0$$

Tuning parameter $\lambda_n = 2n\mu_n$. For $k = 1$:



Definition of the (adaptive) LASSO estimator $\hat{\theta}_{\text{AL}}$

adaptive LASSO estimator (Zou, 2006)

$$\hat{\boldsymbol{\theta}}_{\text{AL}} = \underset{\boldsymbol{\theta}\in\mathbb{R}^k}{\arg\min}\ \|\mathbf{y} - X\boldsymbol{\theta}\|^2 + 2n\mu_n^2 \sum_{i=1}^{k} |\theta_i|/|\hat{\theta}_{\text{OLS},j}| \quad \mu_n > 0$$

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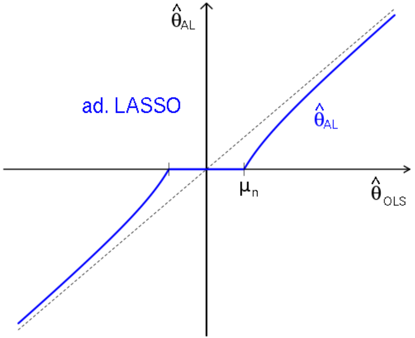

Two regimes for consistency

In terms of model selection consistency, two possible regimes for the tuning parameter μ_n arise.

1. The case $\mu_n \to 0$ and $n^{1/2}\mu_n \to m$, $0 \leq m < \infty$, corresponds to conservative model selection (non-sparsely tuned).
2. The case $\mu_n \to 0$ and $n^{1/2}\mu_n \to \infty$ corresponds to consistent model selection (sparsely tuned).

Remark (estimation consistency).
If $\mu_n \not\to 0$, then $\hat{\theta}_{\text{AL}}$ is not even consistent for $\boldsymbol{\theta}$. Therefore, $\mu_n \to 0$ is a "basic condition".

We will focus on 2 here, also discuss 1 .

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Asymptotics in the consistent case

Zou (2006) "oracle property"

Suppose $X'X/n \to Q > 0$ and $\varepsilon_t \overset{\text{iid}}{\sim} (0, \sigma^2)$.
If $\mu_n \to 0$ and $n^{1/2}\mu_n \to \infty$ and additionally $n^{1/4}\mu_n \to 0$, then

$$n^{1/2}(\hat{\boldsymbol{\theta}}_{\text{AL}} - \boldsymbol{\theta}) \to N(\mathbf{0}, \Sigma_\theta),$$

where Σ_θ is the asymptotic VC-matrix of the restricted LS-estimator based on the unknown true zero restrictions.

Seems to suggest that $\hat{\boldsymbol{\theta}}$ performs as well as if we would know the true zero coefficients of $\boldsymbol{\theta}$.

Questions

- Does this theorem provide meaningful insights? Finite-sample distribution?
- Asymptotic behavior under regime (1) ?
- What if condition $n^{1/4}\mu_n \to 0$ is dropped in (2) ?
- Pointwise vs. uniform consistency rates?
- Properties of confidence intervals?

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We answer these questions within a normal orthogonal linear regression model and address the non-orthogonal case in a simulation study.

Explicit solution in a simple model

- X is non-stochastic $(n \times k)$, $rk(X) = k$.
- $\varepsilon \sim N_n(0, \sigma^2 \mathcal{I}_n)$
- For the theoretical analysis, assume that σ^2 is known and that $X'X$ is diagonal, in particular $X'X = n\mathcal{I}_k$.
- Remove these assumptions for simulation results concerning the finite-sample distribution.

Wlog consider Gaussian location model $y_1, \ldots, y_n \stackrel{\text{iid}}{\sim} N(\theta, 1)$.
Then $\hat{\theta}_{\text{OLS}} = \bar{y}$ with $\hat{\theta}_{\text{OLS}} \sim N(\theta, 1/n)$ and

$$\hat{\theta}_{\text{AL}} = \begin{cases} 0 & \text{if } |\bar{y}| \leq \mu_n \\ \bar{y} - \mu_n^2/\bar{y} & \text{if } |\bar{y}| > \mu_n \end{cases}$$

Selects between restricted $\{N(0,1)\}$ and full model $\{N(\theta, 1) : \theta \in \mathbb{R}\}$

The finite-sample distribution of $\hat{\theta}_{\text{AL}}$

The cdf $F_{n,\theta}(x) = P_{n,\theta}(n^{1/2}(\hat{\theta}_{\text{AL}} - \theta) \leq x)$ of $\hat{\theta}_{\text{AL}}$ is given by

$$\mathbf{1}(n^{1/2}\theta + x \geq 0)\, \Phi\left(z_{n,\theta}^{(2)}(x)\right) + \mathbf{1}(n^{1/2}\theta + x < 0)\, \Phi\left(z_{n,\theta}^{(1)}(x)\right).$$

$z_{n,\theta}^{(2)}(x)$ and $z_{n,\theta}^{(1)}(x)$ are $-(n^{1/2}\theta - x)/2 \pm \sqrt{((n^{1/2}\theta + x)/2)^2 + n\mu_n^2}$.

Φ and ϕ the cdf and pdf of $N(0,1)$, resp.

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$dF_{n,\theta}$ is given by

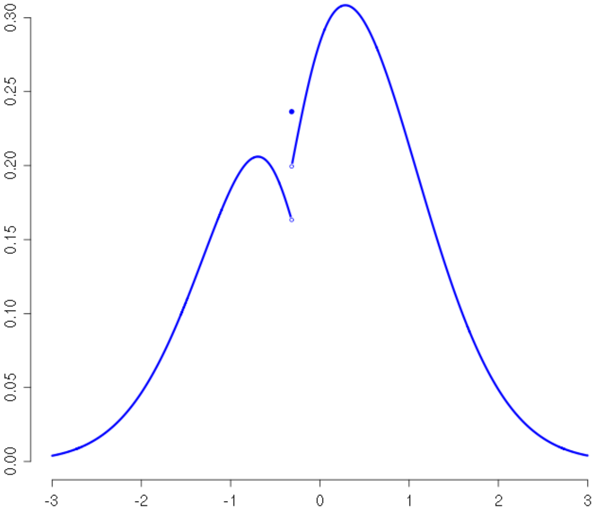
$$\begin{aligned} &\{\, \Phi(n^{1/2}(-\theta + \mu_n)) \; - \; \Phi(n^{1/2}(-\theta - \mu_n)) \,\}\; d\delta_{-n^{1/2}\theta}(x) + \\ &0.5 \times \{\mathbf{1}(n^{1/2}\theta + x > 0)\, \phi\left(z_{n,\theta}^{(2)}(x)\right)\, (1 + t_{n,\theta}(x)) + \\ &\qquad \mathbf{1}(n^{1/2}\theta + x < 0)\, \phi\left(z_{n,\theta}^{(1)}(x)\right)\, (1 - t_{n,\theta}(x))\,\}\; dx \end{aligned}$$

where $t_{n,\theta}(x) := \left(((n^{1/2}\theta + x)/2)^2 + n\mu_n^2\right)^{-1/2}$.

Φ and ϕ the cdf and pdf of $N(0,1)$, resp.

The finite-sample distribution of $\hat{\theta}_{AL}$

$n = 40$, $\theta = 0.05$, $\mu_n = 0.05$

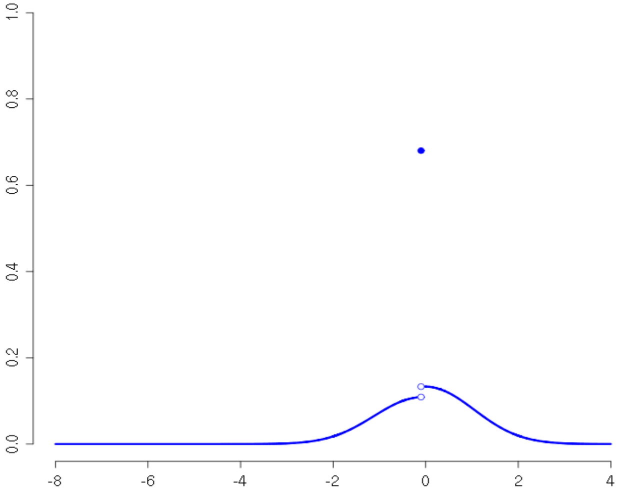


Non-normality??

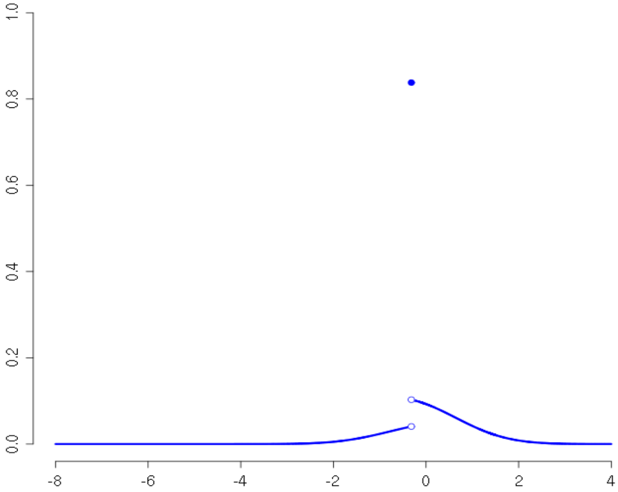
- Finite-sample distribution is highly non-normal.
- Oracle property predicts normality (asymptotically).

The Oracle (fixed-parameter asymptotics)

$n = 1, \qquad \mu_n = n^{-1/3}$ (consistent case)

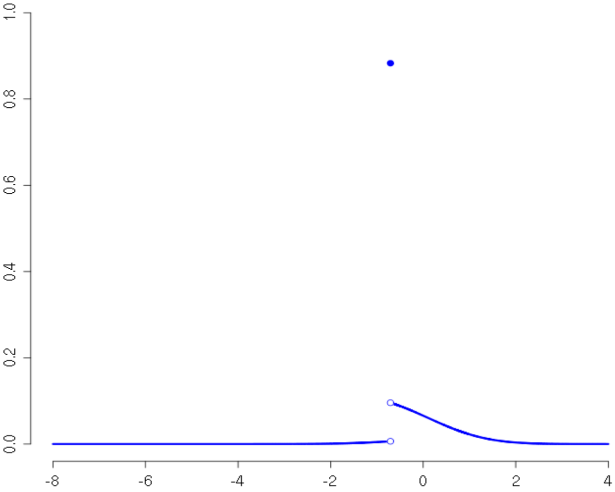


$n = 10, \qquad \mu_n = n^{-1/3}$ (consistent case)



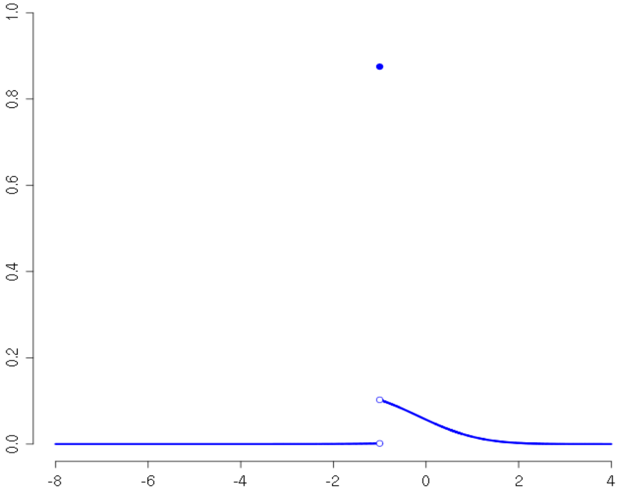
The Oracle (fixed-parameter asymptotics)

$n = 50, \qquad \mu_n = n^{-1/3}$ (consistent case)



The Oracle (fixed-parameter asymptotics)

$n = 100, \qquad \mu_n = n^{-1/3}$ (consistent case)



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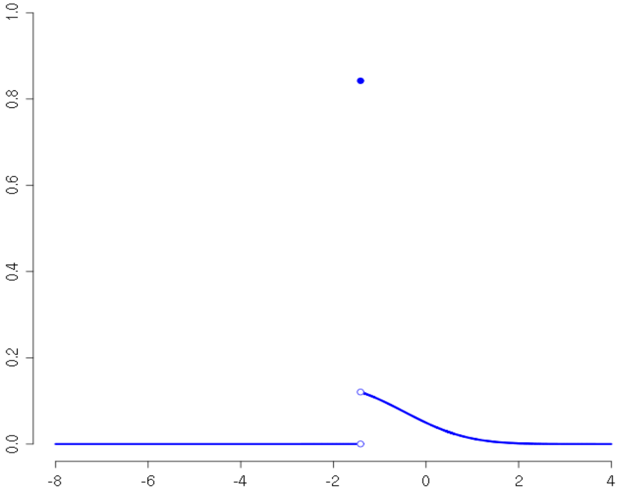
$n = 200, \qquad \mu_n = n^{-1/3}$ (consistent case)



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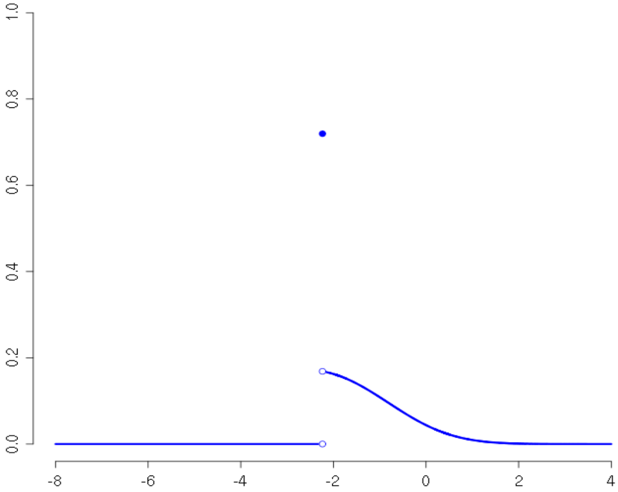
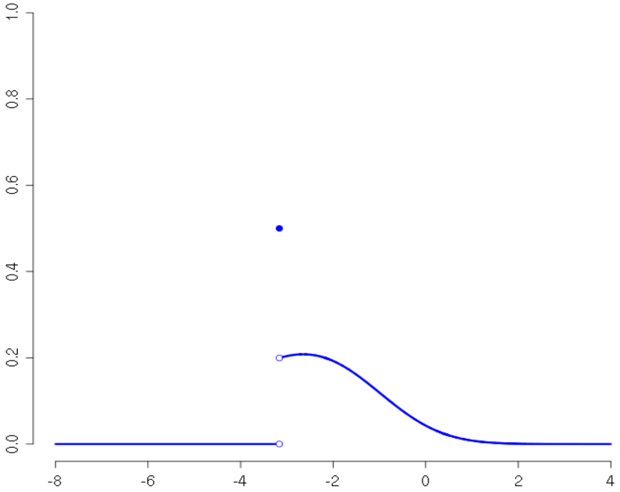
$n = 500, \qquad \mu_n = n^{-1/3}$ (consistent case)

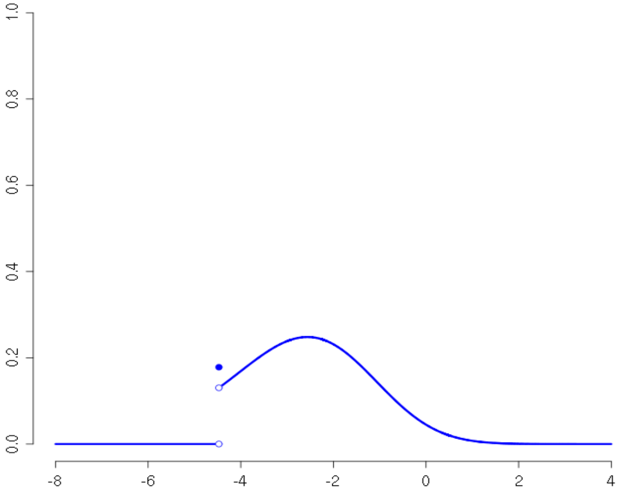


The Oracle (fixed-parameter asymptotics)

$n = 1000, \quad \mu_n = n^{-1/3}$ (consistent case)

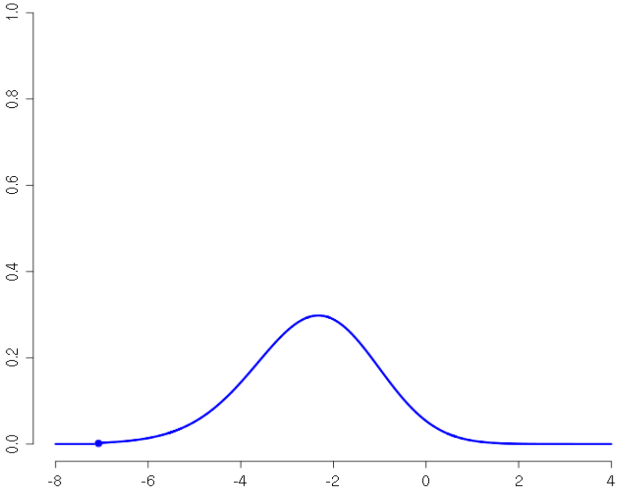


$n = 2000, \quad \mu_n = n^{-1/3}$ (consistent case)

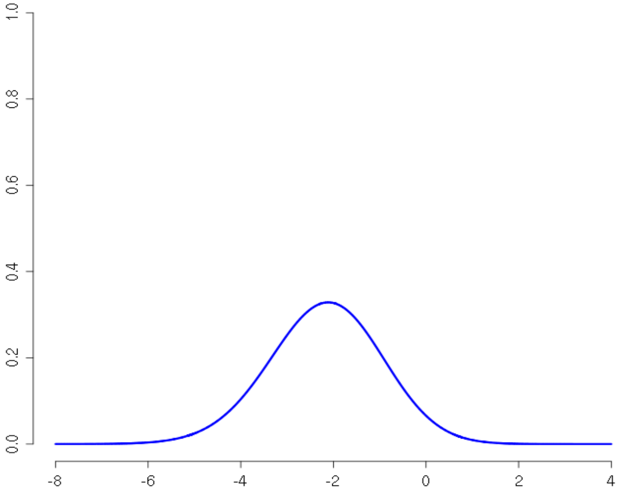


The Oracle (fixed-parameter asymptotics)

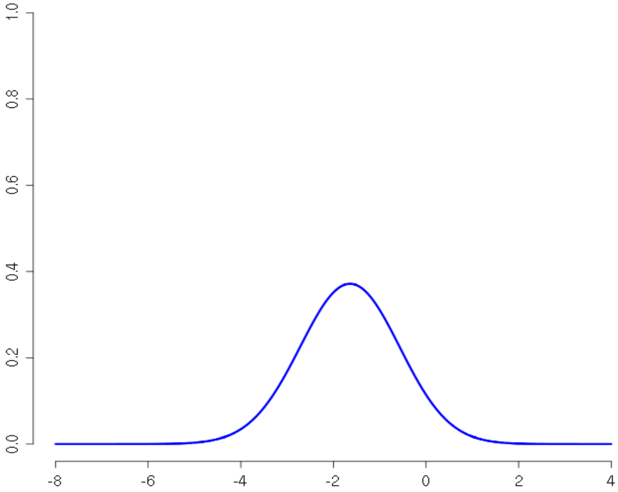
$n = 5000, \quad \mu_n = n^{-1/3}$ (consistent case)



$n = 10^4, \qquad \mu_n = n^{-1/3}$ (consistent case)

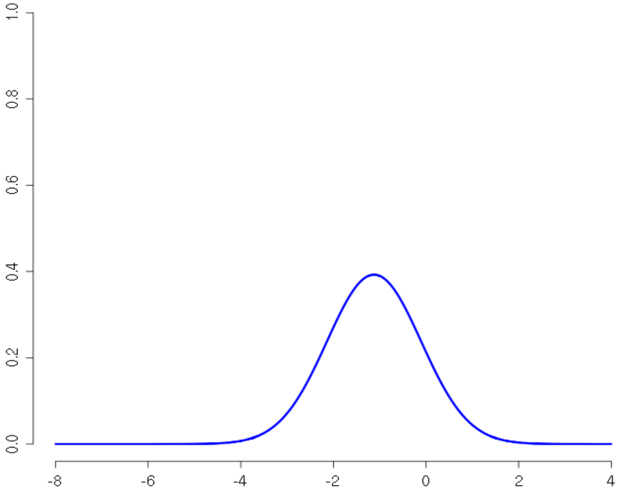


$n = 5 \times 10^4,\ \mu_n = n^{-1/3}$ (consistent case)



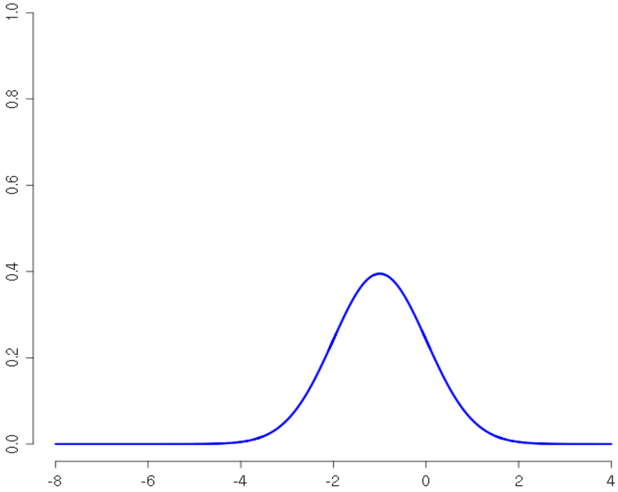
The Oracle (fixed-parameter asymptotics)

$n = 5 \times 10^5,\ \mu_n = n^{-1/3}$ (consistent case)



The Oracle (fixed-parameter asymptotics)

$n = 10^6, \qquad \mu_n = n^{-1/3}$ (consistent case)



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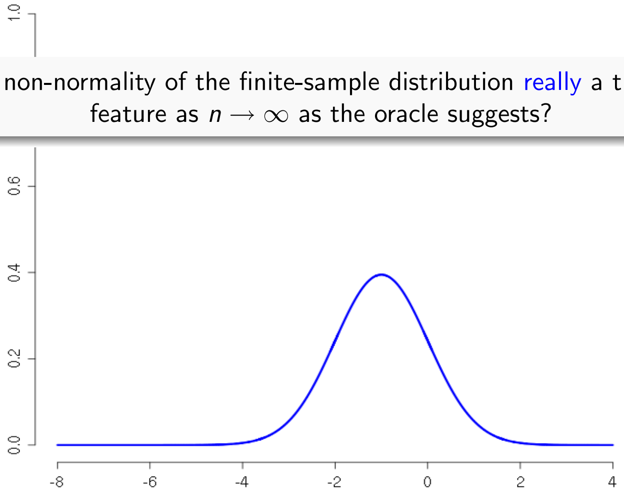

Is the non-normality of the finite-sample distribution really a transient feature as $n \to \infty$ as the oracle suggests?

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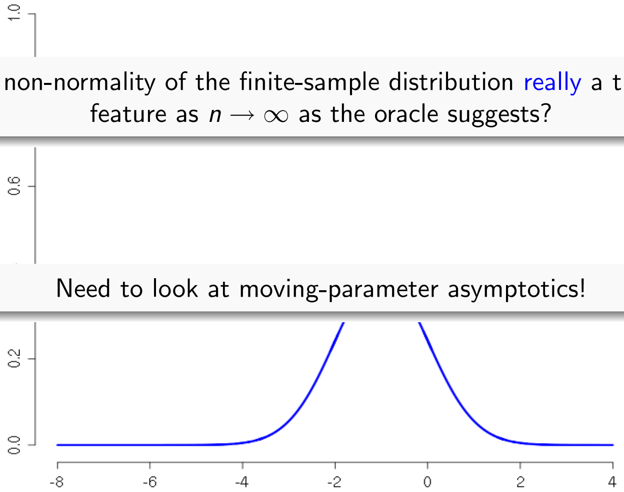

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Need to look at moving-parameter asymptotics!

Moving-parameter asymptotics?

Let underlying parameter θ depend on sample size:

Let $\theta_n \in \mathbb{R}$ be arbitrary, subject only to $\theta_n/\mu_n \to \zeta \in \mathbb{R} \cup \{-\infty, \infty\}$ and $n^{1/2}\theta_n \to \nu \in \mathbb{R} \cup \{-\infty, \infty\}$.

This is not really a restriction since every subsequence of θ_n contains a further subsequence with these properties. Also note that $\zeta \neq 0$ implies $\nu = \pm\infty$.

Moving-parameter asymptotics ②

② Consistent case.

Let $\mu_n \to 0$ and $n^{1/2}\mu_n \to \infty$. Suppose the true parameter $\theta_n \in \mathbb{R}$ satisfies $\theta_n/\mu_n \to \zeta \in \mathbb{R} \cup \{-\infty, \infty\}$ and $n^{1/2}\theta_n \to \nu \in \mathbb{R} \cup \{-\infty, \infty\}$. Then F_{n,θ_n} converges weakly to

- If $0 \leq |\zeta| < \infty$: pointmass at $-\nu$
- If $|\zeta| = \infty$: $\Phi(. + r)$ where $n^{1/2}\mu_n^2/\theta_n \to r$.

Depending on ζ and ν, three possible limits arise.

- Distribution collapses at a point.
- Total mass escapes to $\pm\infty$.
- Limit distribution is normal (shifted!).

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Non-normality persists!!

Illustration: collapsing to pointmass

Example 1: $n = 1$, $\zeta = 0,\ \nu = 2$ $(\mu_n = n^{-1/3},\ \theta_n = 2n^{-1/2})$

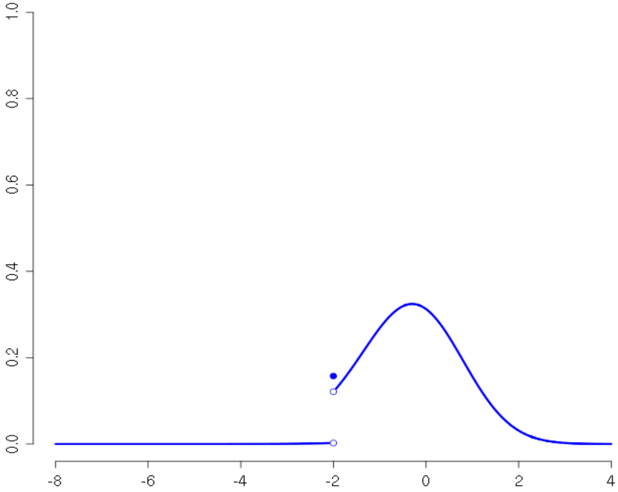
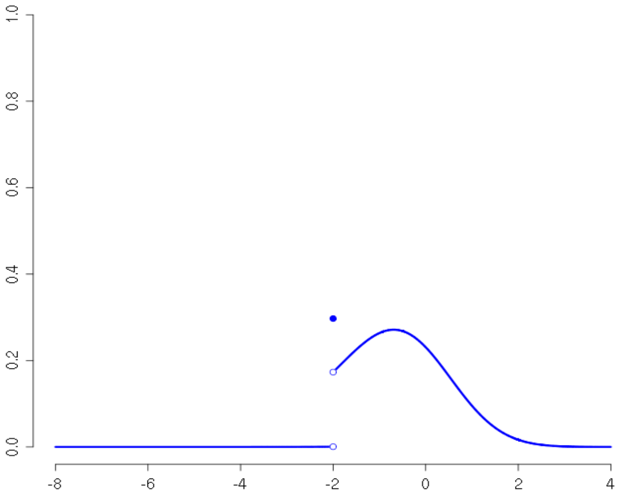


Illustration: collapsing to pointmass

Example 1: $n = 10, \qquad \zeta = 0,\ \nu = 2 \qquad (\mu_n = n^{-1/3},\ \theta_n = 2n^{-1/2})$



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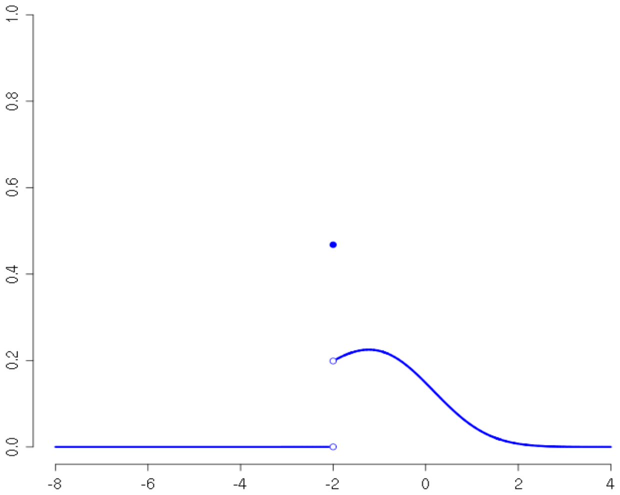

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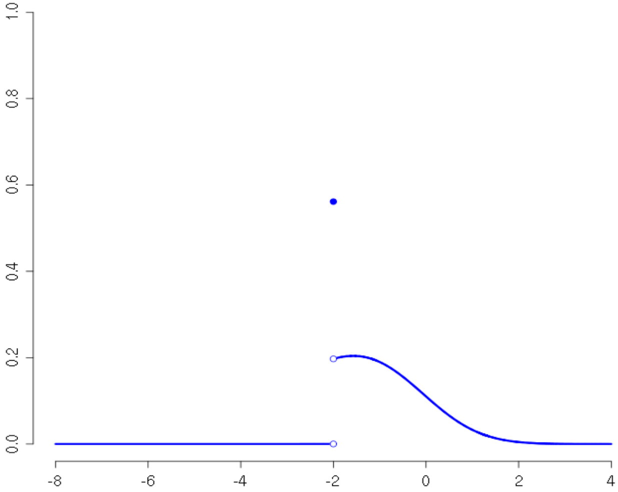

Illustration: collapsing to pointmass

Example 1: $n = 200, \qquad \zeta = 0,\ \nu = 2 \qquad (\mu_n = n^{-1/3},\ \theta_n = 2n^{-1/2})$

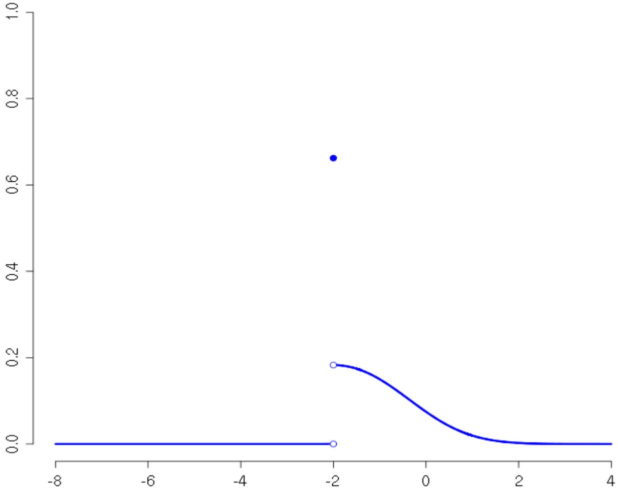

Illustration: collapsing to pointmass

Example 1: $n = 500, \quad \zeta = 0,\ \nu = 2 \quad (\mu_n = n^{-1/3},\ \theta_n = 2n^{-1/2})$

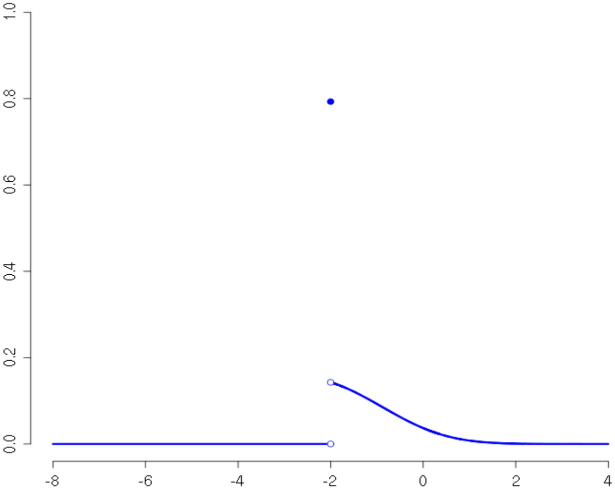

Illustration: collapsing to pointmass

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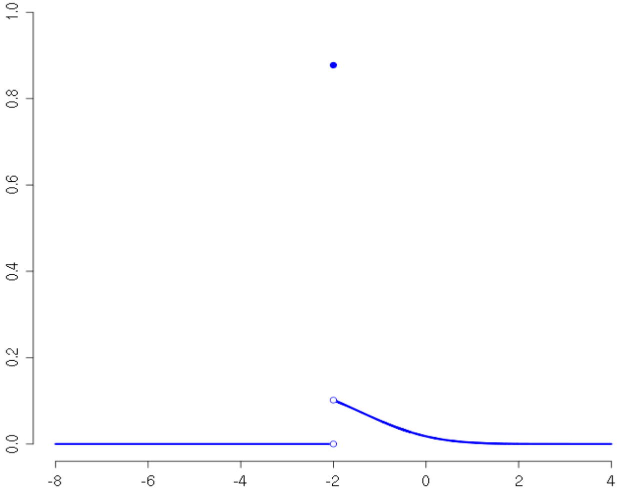

Illustration: collapsing to pointmass

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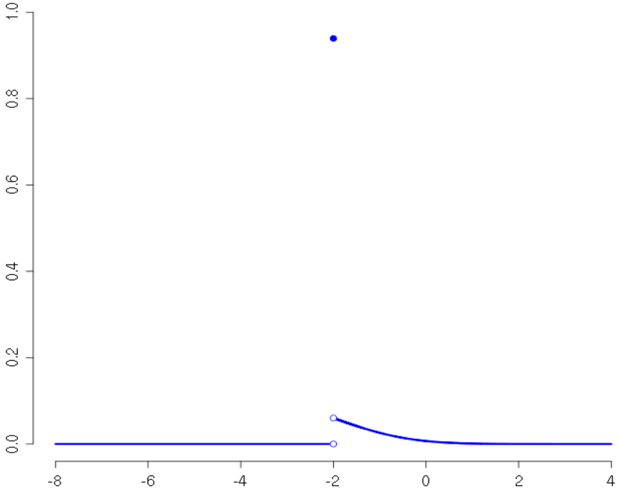

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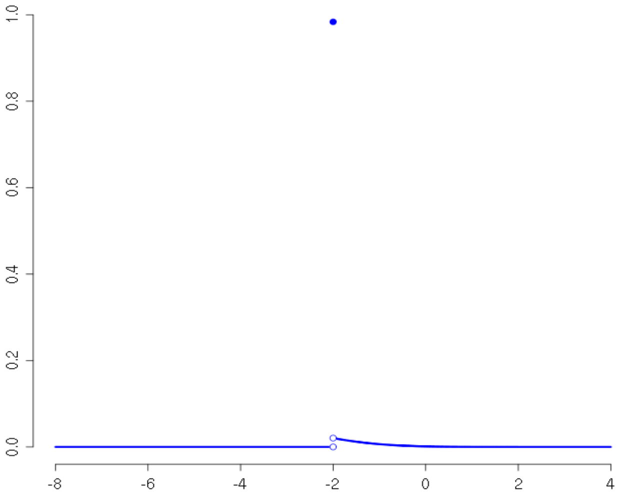


Illustration: collapsing to pointmass

Example 1: $n = 10^4$, $\zeta = 0,\ \nu = 2$ $(\mu_n = n^{-1/3},\ \theta_n = 2n^{-1/2})$

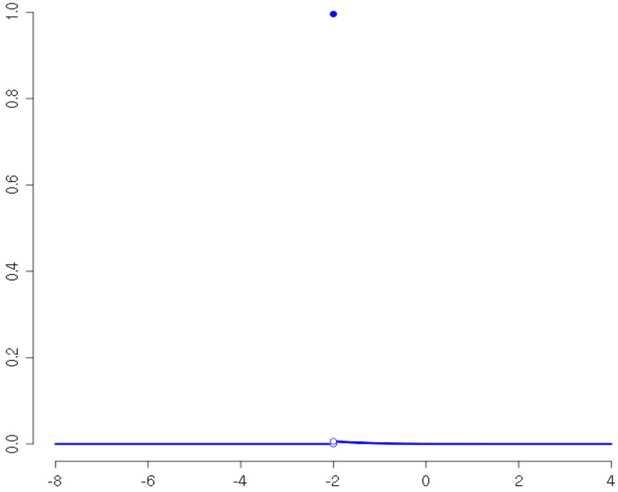
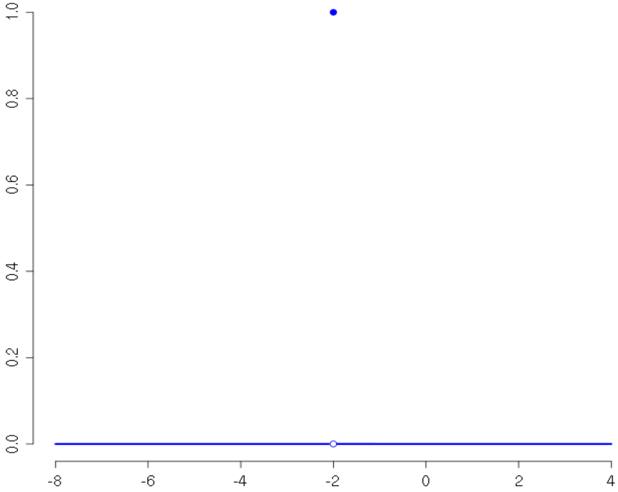


Illustration: collapsing to pointmass

Example 1: $n = 5 \times 10^4$, $\zeta = 0$, $\nu = 2$ $\quad (\mu_n = n^{-1/3},\ \theta_n = 2n^{-1/2})$



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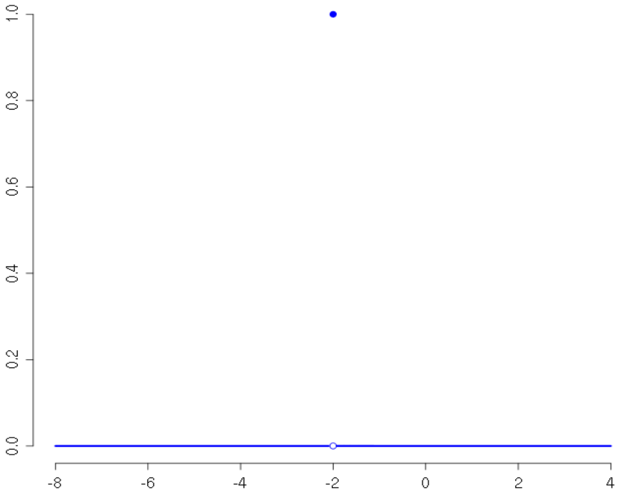

END

Illustration: mass escaping to $-\infty$

Example 2: $n = 1, \qquad \zeta = 1,\ \nu = \infty \qquad (\mu_n = n^{-1/5},\ \theta_n = n^{-1/5})$

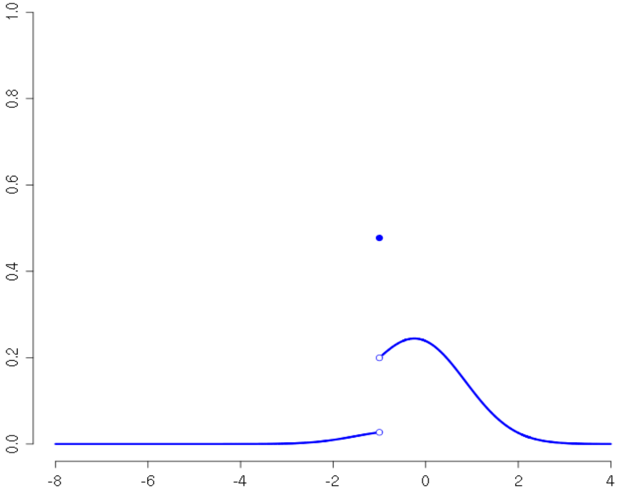

Illustration: mass escaping to $-\infty$

Example 2: $n = 10$, $\zeta = 1,\ \nu = \infty$ $(\mu_n = n^{-1/3},\ \theta_n = n^{-1/5})$

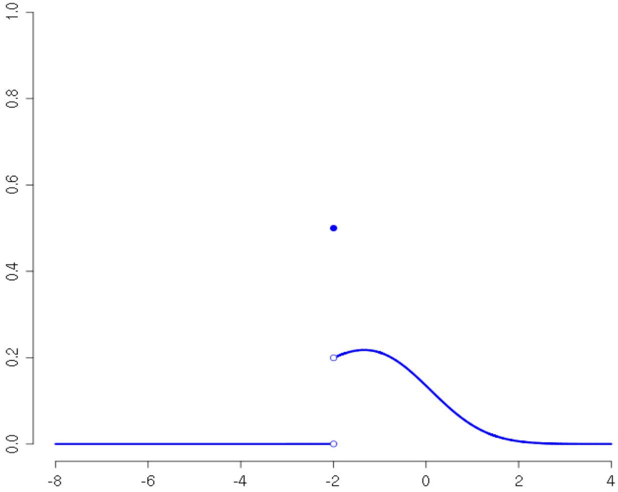

Illustration: mass escaping to $-\infty$

Example 2: $n = 50$, $\zeta = 1,\ \nu = \infty$ $(\mu_n = n^{-1/3},\ \theta_n = n^{-1/5})$

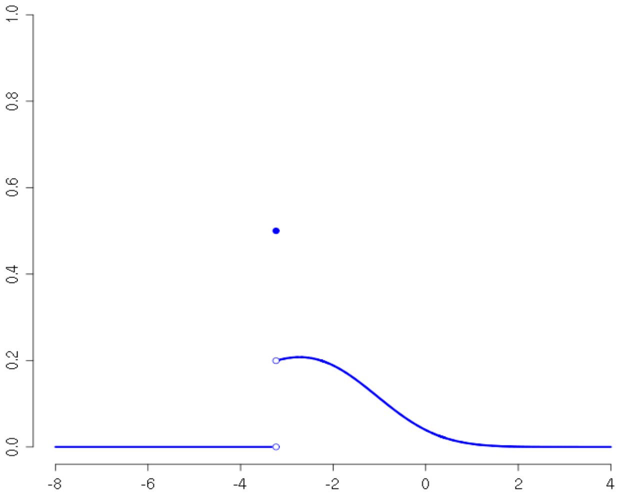

Illustration: mass escaping to $-\infty$

Example 2: $n = 100, \qquad \zeta = 1,\ \nu = \infty \qquad (\mu_n = n^{-1/3},\ \theta_n = n^{-1/5})$

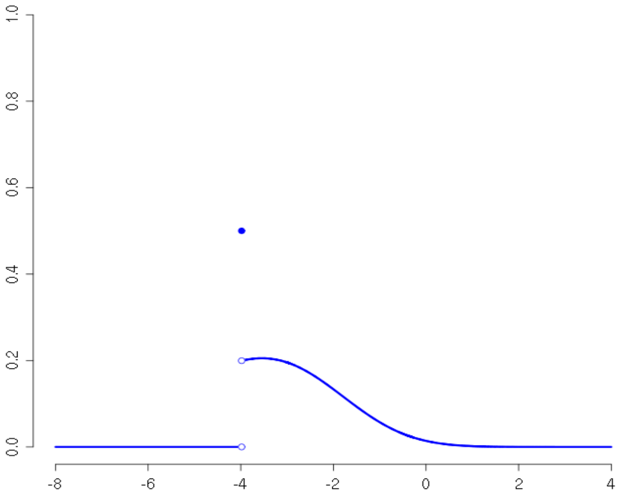

Illustration: mass escaping to $-\infty$

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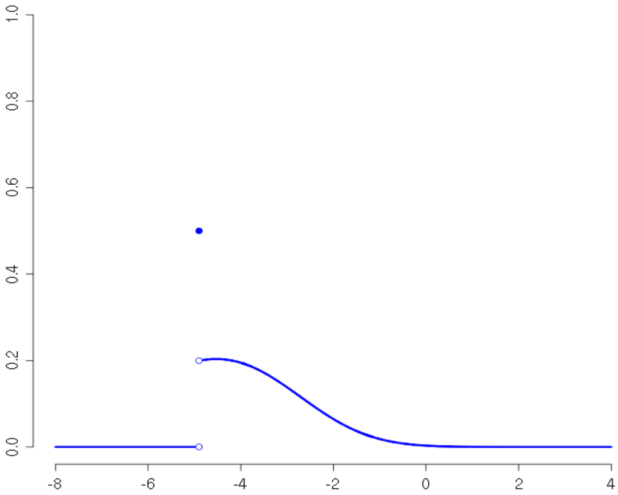

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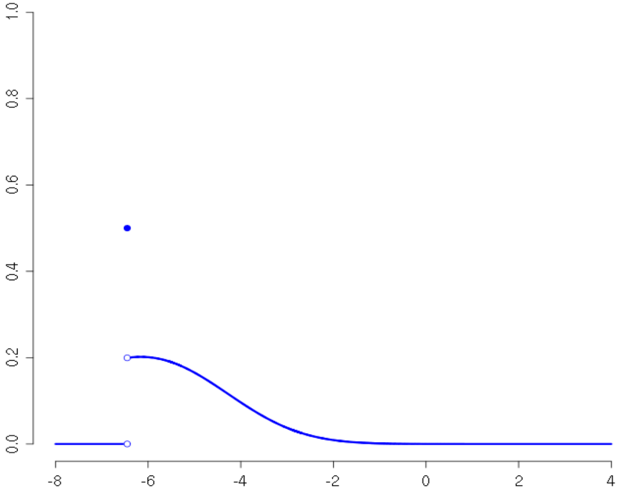

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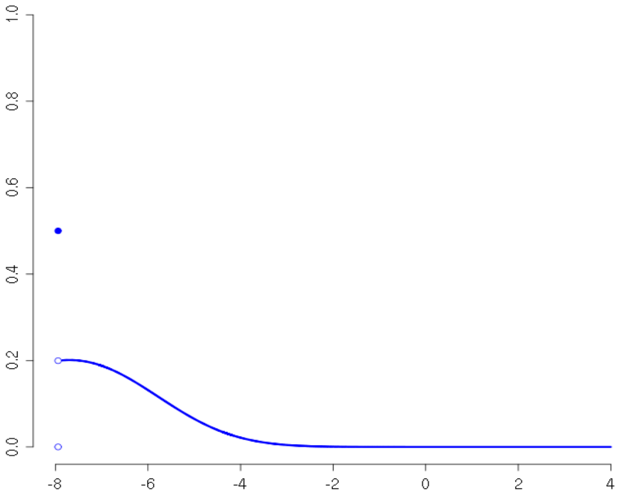

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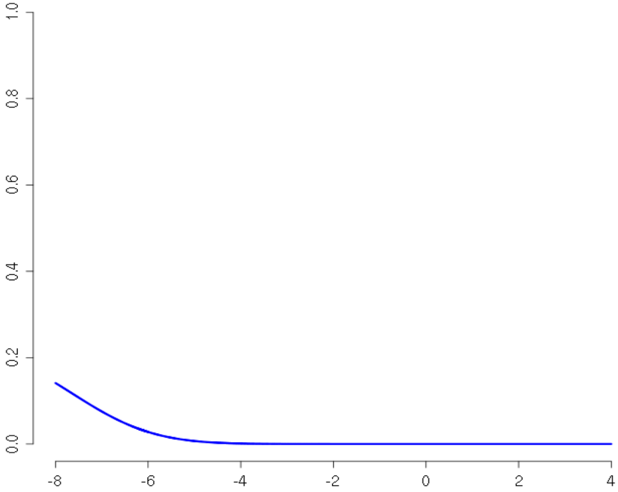

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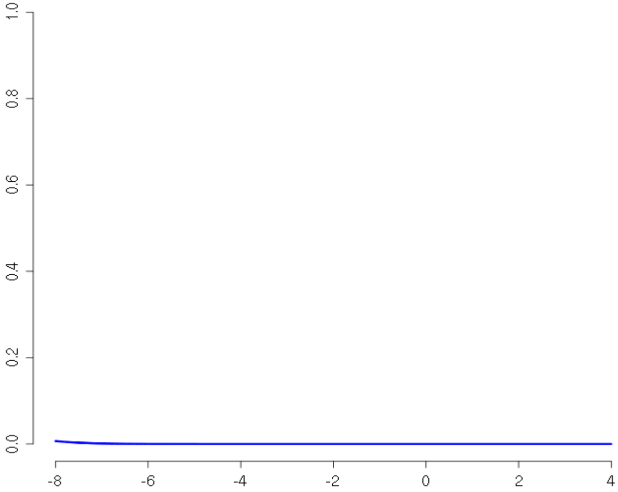

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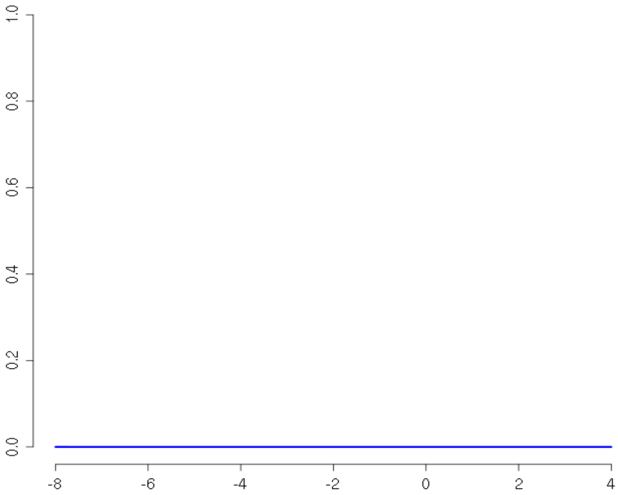

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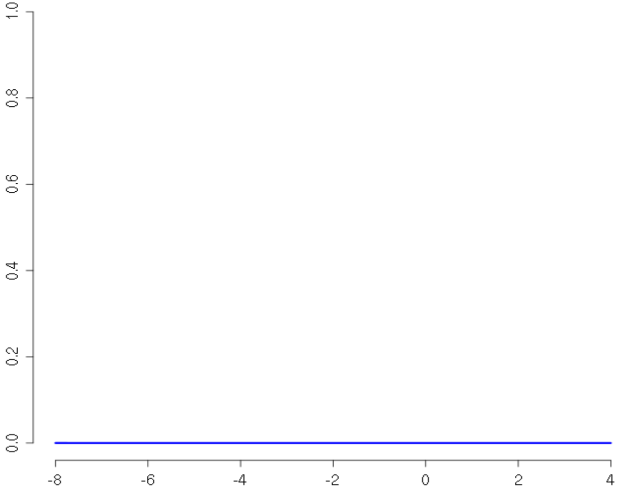

END

Moving-parameter asymptotics ❷ (cont'd)

❷ Consistent case.

Let $\mu_n \to 0$ and $n^{1/2}\mu_n \to \infty$. Suppose the true parameter $\theta_n \in \mathbb{R}$ satisfies $\theta_n/\mu_n \to \zeta \in \mathbb{R} \cup \{-\infty, \infty\}$ and $n^{1/2}\theta_n \to \nu \in \mathbb{R} \cup \{-\infty, \infty\}$. Then F_{n,θ_n} converges weakly to

- If $0 \leq |\zeta| < \infty$: pointmass at $-\nu$
- If $|\zeta| = \infty$: $\Phi(. + r)$ where $n^{1/2}\mu_n^2/\theta_n \to r$.

Zou (pointwise case) ? Above theorem implies that

$$F_{n,\theta}(x) \to \begin{cases} \mathbf{1}(x \geq 0) & \theta = 0 \quad (\implies \zeta, \nu = 0) \\ \Phi(x + \rho/\theta) & \theta \neq 0 \quad (\implies |\zeta| = \infty), \; n^{1/2}\mu_n^2 \to \rho \end{cases}$$

Remark: $\rho = 0 \iff n^{1/4}\mu_n \to 0$.

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Uniform consistency and alternative scaling 2

- Adaptive LASSO has in a uniform sense a rate of convergence that is slower than $n^{1/2}$.
- The "correct" uniform rate can be shown to be μ_n^{-1}.
- In a moving-parameter framework, the asymptotic distribution of $\mu_n^{-1}(\hat{\theta}_{\text{AL}} - \theta)$ collapses to pointmass.

Above theorems reflect that

$$\hat{\theta}_{\text{AL}} - \theta = \text{"BIAS"} + \text{"FLUCTUATION"}$$

- "BIAS" is $O(n^{-1/2})$ in a pointwise sense but is only $O(\mu_n)$ in a uniform sense, whereas
- "FLUCTUATION" is always of order $n^{-1/2}$.

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1. Conservative case.

Let $\mu_n \to 0$ and $n^{1/2}\mu_n \to m$, $0 \le m < \infty$. Suppose the true parameter $\theta_n \in \mathbb{R}$ satisfies $n^{1/2}\theta_n \to \nu \in \mathbb{R} \cup \{-\infty, \infty\}$. Then F_{n,θ_n} converges weakly to

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Note: Asymptotic distributions are the same as finite-sample distribution, except that $n^{1/2}\theta_n$ and $n^{1/2}\mu_n$ have settled down to their limiting values, capturing finite-sample behavior very well.

Moving parameter asymptotics ❶ (cont'd)

- $\hat{\theta}_{\text{AL}}$ is now uniformly $n^{1/2}$-consistent.
- Fixed-parameter asymptotics: previous theorem implies that $F_{n,\theta}(x)$ converges to
  - $\mathbf{1}(x \geq 0)\,\Phi\left(\frac{x}{2} + \sqrt{(\frac{x}{2})^2 + m^2}\right) + \mathbf{1}(x < 0)\,\Phi\left(\frac{x}{2} - \sqrt{(\frac{x}{2})^2 + m^2}\right)$
    if $\theta = 0$ $(\nu = 0)$
  - $\Phi(x)$ if $\theta \neq 0$ $(|\nu| = \infty)$
- Fixed-parameter asymptotic distributions are also non-normal, capturing behavior the finite-sample distributions to some extent (no oracle here).

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Other PLSEs

Results are similar for hard-thresholding, soft-thresholding (LASSO), and SCAD estimator. (Pötscher & Leeb, 2009).

- Identical results in terms of (uniform) consistency.
- Analogous (asymptotic) distributional results.

Lengths of confidence sets based on PLSEs

Let $C_n = [\hat{\theta} - a_n, \hat{\theta} + a_n]$ with $\inf_{\theta \in \mathbb{R}} P(\theta \in C_n) \geq \delta$.

- For each $n \in \mathbb{N}$, we have

$$a_{n,H} > a_{n,\mathrm{AL}} > a_{n,L} > a_{n,\mathrm{OLS}} \quad \text{for a given } \delta > 0$$

- Asymptotically, the following holds.

1. Conservative case. All quantities are of the same order $n^{-1/2}$.

$$a_{n,H} \sim a_{n,\mathrm{AL}} \sim a_{n,L} \sim a_{\mathrm{OLS}}$$

2. Consistent case. $a_{n,H}$, $a_{n,L}$, and $a_{n,A}$ are one order of magnitude larger than $a_{n,\mathrm{OLS}}$.

$$a_H / a_{\mathrm{OLS}} \sim a_{\mathrm{AL}} / a_{\mathrm{OLS}} \sim a_L / a_{\mathrm{OLS}} \sim n^{1/2} \mu_n \to \infty$$

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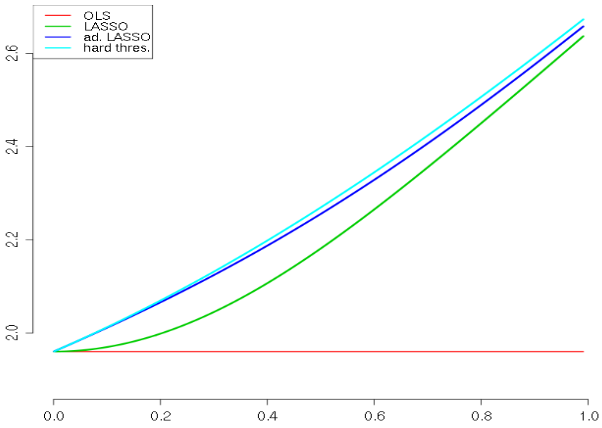
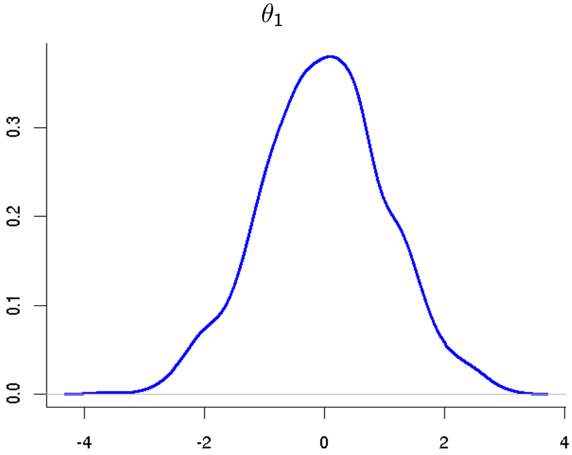
Plot of $n^{1/2}a_n$ against $n^{1/2}\mu_n$ for $\delta = 0.95$.



Simulations - remove orthogonality assumption

$k = 4$, $n = 200$, $\theta = (3, 1.5, 0, 0)' + 2/n^{1/2}(0, 0, 1, 1)'$, $X'X = n\Omega$ with $\Omega_{ij} = 0.5^{|i-j|}$, 1000 simulations

- $\mu_n = n^{-1/3}$



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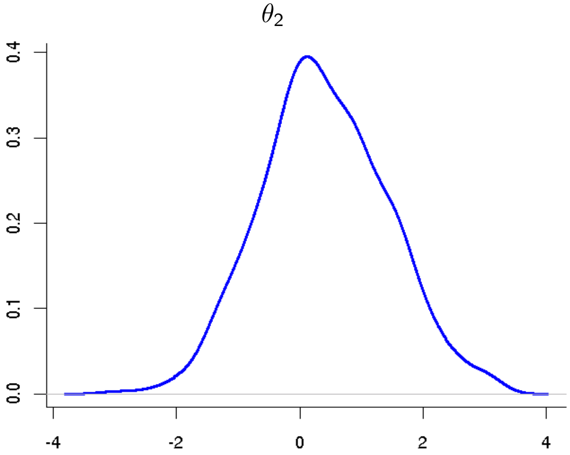

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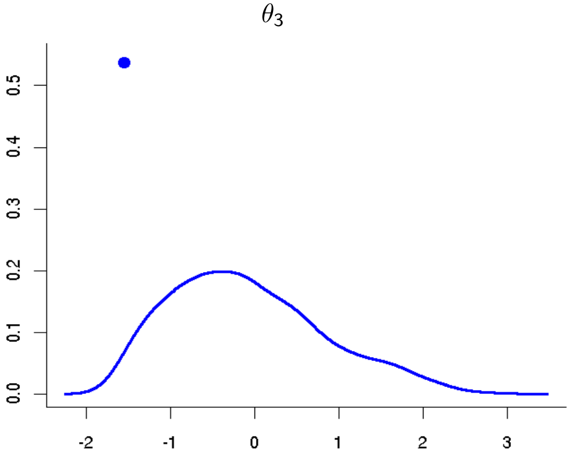

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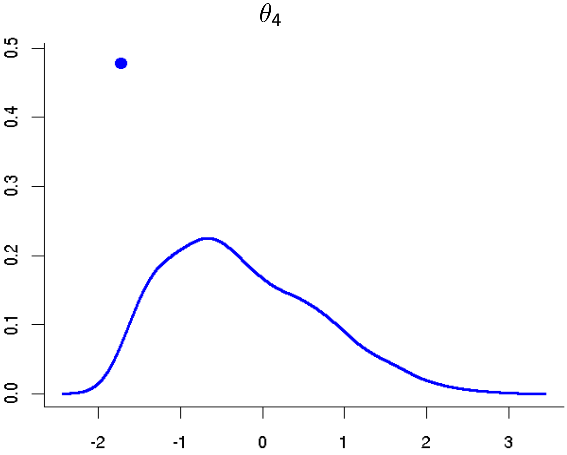

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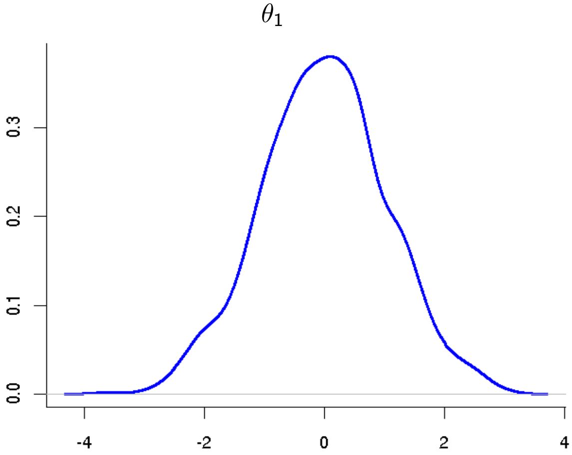
- Choose μ_n through cross-validation.



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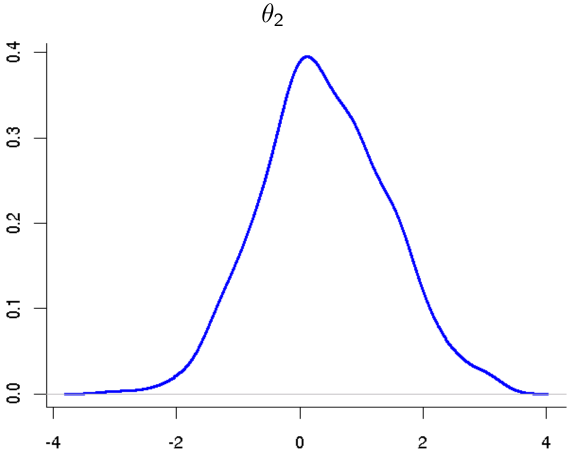

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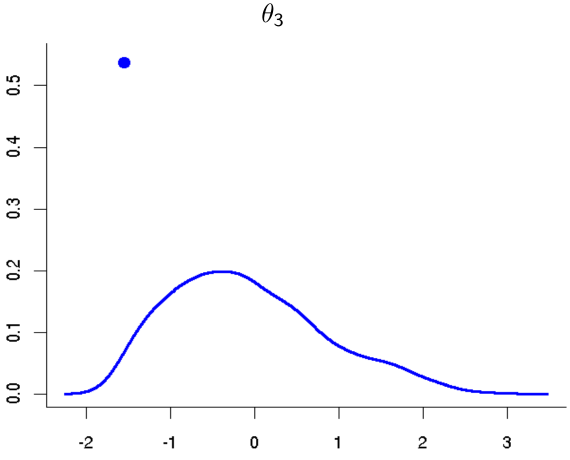

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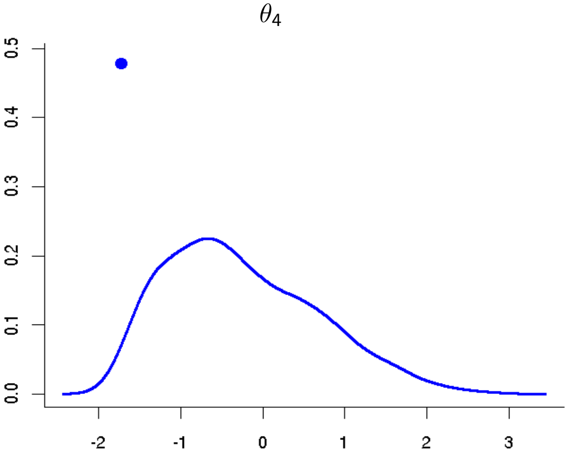

Conclusions

- The finite-sample distribution of the adaptive LASSO estimator and other PLSEs are highly non-normal.
- Non-normality persists in large samples. This can be seen through a "moving-parameter" asymptotic framework.
- Fixed-parameter asymptotics (as underlying the oracle-property) paint a misleading picture of the performance of the estimator due to the non-uniformity of these results.
- Confidence intervals in the consistent case are larger by one order of magnitude compared to the unpenalized estimator.
- NOT a critisim on PLSEs per se, but relying on fixed-parameter asymptotics in this context is dangerous.

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