Cointegration with infinite variance noise

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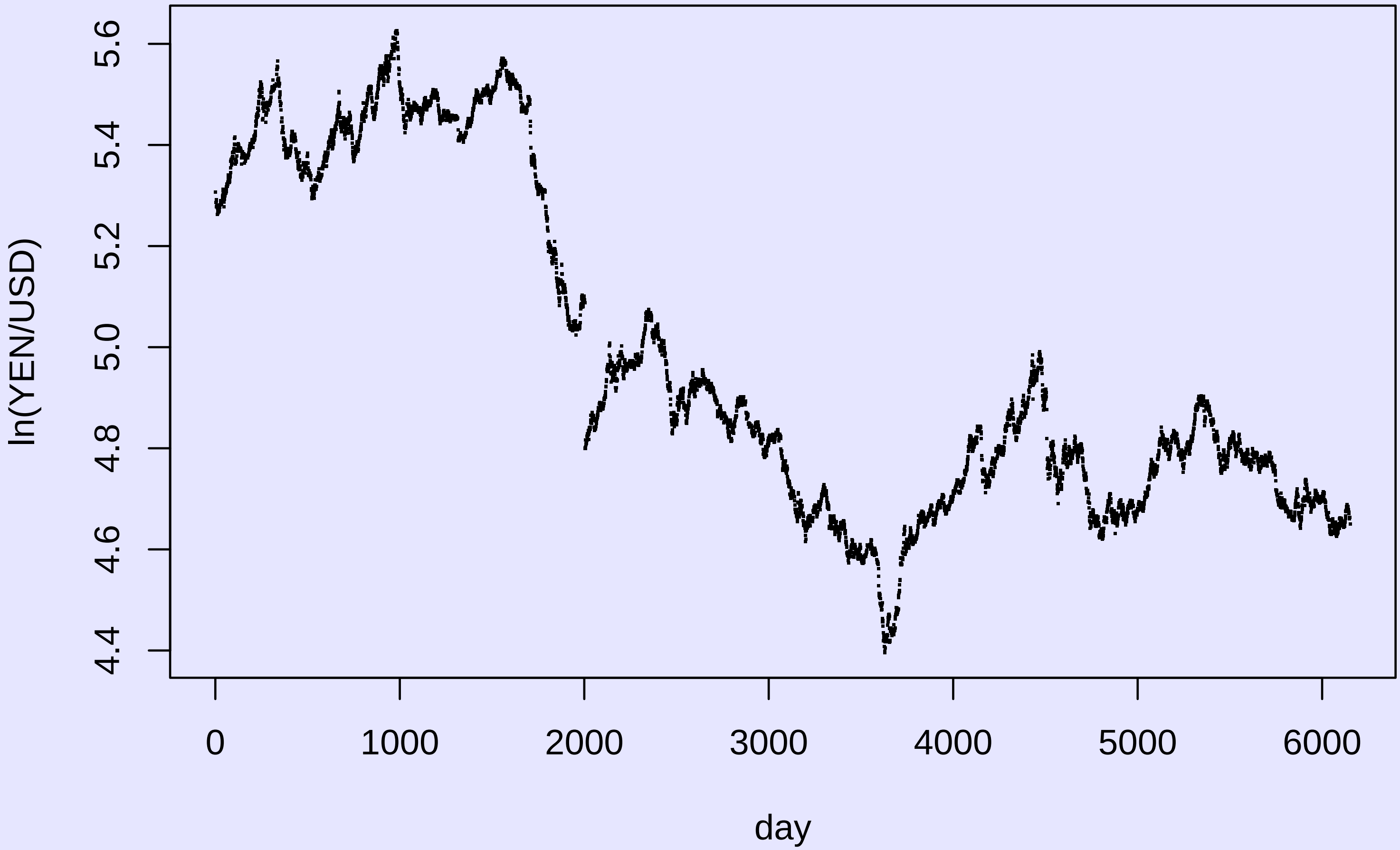
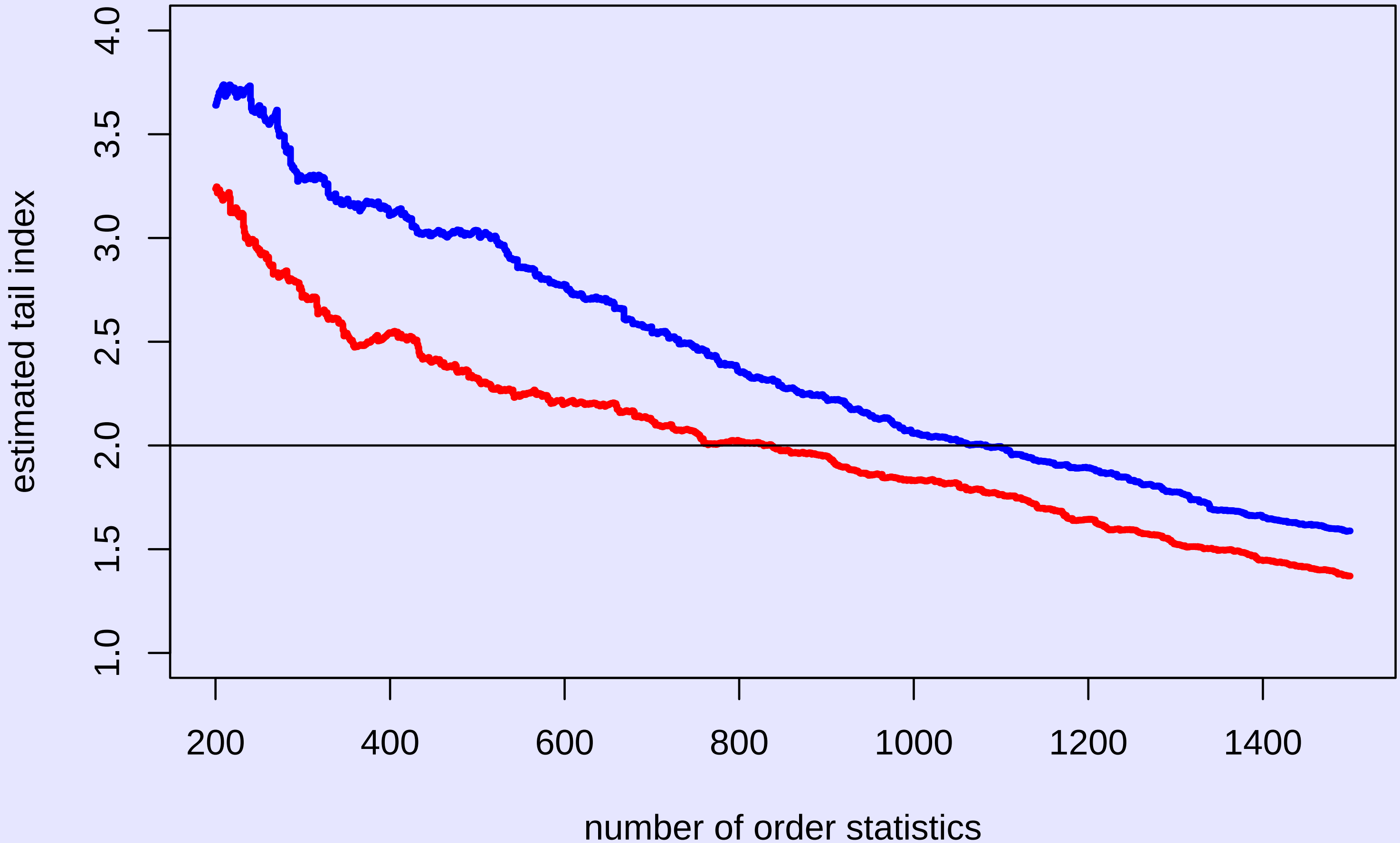
1. INTRODUCTION

Time series analysis with heavy tails

- Mandelbrot (1963, 1967) and Fama (1965) observed that distributions of stock returns are often heavy tailed with possibly infinite variance.
- Since that time, there has been extensive work on examining the plausibility of the infinite variance model.
- Philosophical/modeling question: Are variances infinite or finite with stochastic heteroscedasticity?



Yen per US dollar exchange rate: Dec 1978 to Apr 2005



Hill estimates for returns: lower tail = red, upper tail = blue.

- A partial list of research in this area includes:
  - Stationary time series: Davis & Resnick (1995, 1996), Davis, Knight & Liu (1992), Anderson & Meerschaert (1997).
  - Unit root testing: Chan & Tran (1989), Knight (1989), Phillips (1990), Rachev, Mittnik & Kim (1998), Hasan (2001), Ahn, Fotopoulos & He (2001), Martins (2009), Samarakoon & Knight (2009).
  - Cointegration testing: Caner (1998), Paulauskas & Rachev (1998), Xiao (2009).
  - Applications: Koedijk and Kool (1992), Falk and Wang (2003), Charemza, Hristova & Burridge (2005), Kirman and Teyssière (2005).

What is cointegration?

- A univariate stochastic process $\{X_t\}$ is **integrated** if it is non-stationary but its first differences $\nabla X_t = X_t - X_{t-1}$ are stationary.
- If $\{\boldsymbol{X}_t\}$ is a vector process whose elements are all integrated then it is cointegrated if $\{\boldsymbol{a}^\top \boldsymbol{X}_t\}$ is stationary for some $\boldsymbol{a} \neq 0$ (called a cointegration vector).
- Economic interpretation: individual variables behave like random walks but are collectively in equilibrium.

Example: Long run PPP

$$\text{Is } \ln(\text{exchange rate}_t) - \ln(\text{CPI}_t^{(1)}/\text{CPI}_t^{(2)}) \text{ stationary?}$$

Testing for cointegration: Two basic approaches:

(1) Find an estimator $\widehat{\boldsymbol{a}}$ of $\boldsymbol{a}$ (for example, using regression) and test if $\{\widehat{\boldsymbol{a}}^\top \boldsymbol{X}_t\}$ is stationary.

(2) Assume a parametric model (for example, VAR) for $\{\boldsymbol{X}_t\}$ and test for cointegration within that model.

- We will consider the second approach using a VAR model.
- Assume the VAR(k) model written in its error correction form

$$\nabla \boldsymbol{X}_t = \Pi \boldsymbol{X}_{t-k} + \Phi_1 \nabla \boldsymbol{X}_{t-1} + \cdots + \Phi_{k-1} \boldsymbol{X}_{t-k+1} + \boldsymbol{\varepsilon}_t$$

- We will assume that the components of $\{\boldsymbol{\varepsilon}_t\}$ have infinite variance.

- Define the cointegration space of $\{\boldsymbol{X}_t\}$:

$$\mathcal{C} = \{\boldsymbol{a} : \{\boldsymbol{a}^\top \boldsymbol{X}_t\} \text{ is stationary}\}$$

  $\mathcal{C}$ is simply the row space of Π.

  – If $\Pi = 0$ then there is no cointegration.

  – If Π has full rank then $\{\boldsymbol{X}_t\}$ is stationary.

- Cointegration rank is determined essentially by finding good lower rank approximations to an unconstrained estimator of Π.

  – Start by testing $H_0 : \Pi = 0$ (no cointegration).

- Johansen (1988, 1991, ...) and others have developed extensive asymptotic distribution theory in the finite variance case.

Question: How to extend to the infinite variance case?

- Finite variance procedures work quite well but the asymptotic theory is complicated (Caner, 1998; Paulauskas & Rachev, 1998).
- For simplicity, we will consider component-by-component M-estimators of the parameters in the model.
- Define Y_t to be an arbitrary component of ∇X_t.
- Our M-estimators minimize

  $$\sum_{t=k+1}^{n} \rho(Y_t - \boldsymbol{X}_{t-k}^{\top}\boldsymbol{\pi} - \nabla X_{t-1}^{\top}\boldsymbol{\phi}_1 - \cdots - \nabla X_{t-k+1}^{\top}\boldsymbol{\phi}_{k-1})$$

  over some appropriate space where ρ is a convex function increasing slower than x^2.
- These estimators can be “stacked” to give estimators of Π, $\Phi_1, \cdots, \Phi_{k-1}$.

2. ASYMPTOTICS

- Start with the Granger representation of $\{\boldsymbol{X}_t\}$:

$$\boldsymbol{X}_t = \boldsymbol{X}_0 + A\left\{B^\top(I - \Phi_1 - \cdots - \Phi_{k-1})A\right\}^{-1} B^\top \sum_{s=1}^{t} \boldsymbol{\varepsilon}_s + \boldsymbol{\zeta}_t$$

  where

  - $\{\boldsymbol{\zeta}_t\}$ stationary,
  - $B^\top \Pi = \Pi A = 0$ for maximal rank matrices A and B.

- Assume that $\{B^\top \boldsymbol{\varepsilon}_t\}$ lie in the domain of attraction of an operator stable distribution:

$$\Lambda_n^{-1} \sum_{t=1}^{n} B^\top \boldsymbol{\varepsilon}_t \xrightarrow{d} \boldsymbol{V} \sim P_E$$

  for some E and some sequence of matrices $\{\Lambda_n\}$.

What are operator stable laws?

- Limits of partial sums are operator stable laws P_E, where the index E is a matrix.
- If $\boldsymbol{U}_1, \cdots, \boldsymbol{U}_n$ are i.i.d. P_E then for some $\boldsymbol{b}_n$,

$$n^{-E} \sum_{i=1}^{n} \boldsymbol{U}_i - \boldsymbol{b}_n \sim P_E$$

  where

$$n^{-E} = \exp[-E \ln(n)] = \sum_{k=0}^{\infty} \frac{(-1)^k \ln^k(n) E^k}{k!}.$$

- Canonical form of the characteristic function was given by Sharpe (1969).
- Applications: Meerschaert & Scheffler (2000, 2001).

- The matrix E has eigenvalues $\lambda_1, \cdots, \lambda_p$ with $\mathrm{Re}(\lambda_j) \geq 1/2$ — these play the role of $1/\alpha$ where α is the stable index:
  - If $\mathrm{Re}(\lambda_j) > 1/2$ for all j then P_E is an infinite variance operator stable law.
  - $\mathrm{Re}(\lambda_j) = 1/2$ corresponds to a Gaussian component, which is independent of the infinite variance components.
- P_E must *not* be concentrated on a lower dimensional hyperplane.
  - A lower dimensional projection of an operator stable distribution is not necessarily operator stable.
  - But ... one-dimensional projections have potentially different tail indices.

- An i.i.d. sequence $\{\boldsymbol{U}_t\}$ is in the domain of attraction of P_E if there exists a sequence of matrices $\{\Delta_n\}$ and vectors $\{\boldsymbol{b}_n\}$ such that

$$\Delta_n^{-1} \sum_{t=1}^{n} \boldsymbol{U}_t - \boldsymbol{b}_n \xrightarrow{d} P_E.$$

  - $\{\Delta_n\}$ is regularly varying in the following sense:

$$\lim_{n \to \infty} \Delta_{\lfloor sn \rfloor} \Delta_n^{-1} = s^E \quad \text{for each } s > 0.$$

  - For any set D bounded away from $\mathbf{0}$, we have

$$\lim_{n \to \infty} nP(\Delta_n^{-1} \boldsymbol{U}_t \in D) = \phi(D)$$

    for some measure ϕ.

Example: $\{X_i\}$, $\{Y_i\}$ i.i.d. sequences with $E(X_i) = 0$, $E(X_i^2) = 1$, $Y_i \sim$ Cauchy.

- Define

$$\mathbf{U}_i = \begin{pmatrix} X_i + Y_i \\ X_i - Y_i \end{pmatrix}.$$

- Elements of $\mathbf{U}_i$ are in the domain of attraction of a Cauchy distribution and

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{U}_i \xrightarrow{d} \begin{pmatrix} Y_0 \\ -Y_0 \end{pmatrix}$$

  where $Y_0 \sim$ Cauchy.

- The limiting distribution is concentrated on a one-dimensional subspace of R^2.

- Now look at operator stable limiting distribution.
- Define

$$\Delta_n = \begin{pmatrix} n^{1/2} & n \\ n^{1/2} & -n \end{pmatrix}$$

- Then

$$\Delta_n^{-1} \sum_{i=1}^{n} \mathbf{U}_i \xrightarrow{d} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}$$

  where X_0 and Y_0 are independent, $X_0 \sim \mathcal{N}(0, 1)$ and $Y_0 \sim$ Cauchy.
- For this example,

$$E = \begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{pmatrix}$$

  and E has eigenvalues $1/2$ and 1.

Asymptotics for M-estimation

- Asymptotic distribution theory for estimators of Π combines the techniques used in
  - Davis *et al.* (1992) for stationary AR processes,
  - Knight (1989, 1991) for the unit root AR(1) process, and
  - Samarakoon & Knight (2009) for general unit root tests.
- The asymptotics depend on whether we do unconstrained minimization or project onto $\mathcal{C}^{\perp}$.
  - unconstrained — point process (i.e. non-standard) asymptotics.
  - projections — more classical asymptotics involving an operator stable process and a Brownian motion.

What are the regularity conditions?

- $\{B^\top \boldsymbol{\varepsilon}_t\}$ are in the domain of attraction of an operator stable law with index E whose eigenvalues are greater than $1/2$ with $\boldsymbol{b}_n = 0$;

- ρ is a convex function with derivatives $\psi = \rho'$ and $\psi' = \rho''$ satisfying

$$\begin{aligned} |\psi(x+y) - \psi(x)| &\leq K_1 |y|^{\delta_1} \quad \text{and} \\ |\psi'(x+y) - \psi'(x)| &\leq K_2 |y|^{\delta_2} \end{aligned}$$

  where $\delta_1 > \max\{2(\alpha-1)/\alpha, 0\}$, $\delta_2 > 0$, and K_1, K_2 are positive constants;

- $E[\psi(\varepsilon_{ti})] = 0$, $E[\psi^2(\varepsilon_{ti})] < \infty$, and $0 < E[\psi'(\varepsilon_{ti})] < \infty$ where $\boldsymbol{\varepsilon}_t = (\varepsilon_{t1}, \cdots, \varepsilon_{tp})^\top$.

- Look at asymptotic behaviour of $\{\boldsymbol{X}_t\}$ on $\mathcal{C}^{\perp}$.
- Define partial sum process $\boldsymbol{S}_n$ as follows:

$$\begin{aligned} \boldsymbol{S}_n(u) &= \Delta_n^{-1} A^{\top} \boldsymbol{X}_{\lfloor nu \rfloor} \\ &= \Lambda_n^{-1} \sum_{s=1}^{\lfloor nu \rfloor} B^{\top} \boldsymbol{\varepsilon}_s + o_p(1) \end{aligned}$$

where

$$\Delta_n^{-1} = \Lambda_n^{-1} \left\{ B^{\top} (I - \Phi_1 - \cdots - \Phi_{k-1}) A \right\}$$

- $\boldsymbol{S}_n \xrightarrow{f-d} \boldsymbol{S}_E$, a operator stable Lévy process.

Results: Focus on estimation of Π with rows projected to $\mathcal{C}^\perp$; these should be close to 0.

- If $\widehat{\Pi}_n$ is the "stacked" estimator then

  $$n^{1/2}\Delta_n^\top A^\top \widehat{\Pi}_n^\top \xrightarrow{d} \left(\int_0^1 A^\top \boldsymbol{S}_E(s)\boldsymbol{S}_E^\top(s) A\, ds\right)^{-1}\left(\int_0^1 A^\top \boldsymbol{S}_E(s)\, d\boldsymbol{W}^\top(s)\right)\Gamma^{-1}$$

  where

  - columns of A are an orthonormal basis for $\mathcal{C}^\perp$;
  - $\boldsymbol{W}$ is a zero-mean Gaussian process independent of S_E) with

    $$E[\boldsymbol{W}(s_1)\boldsymbol{W}^\top(s_2)] = \min(s_1, s_2)\Sigma, \;\; \Sigma = \Big(\mathrm{Cov}[\psi(\varepsilon_{ti}), \psi(\varepsilon_{tj})]\Big);$$

  - $\Gamma = \mathrm{diag}(E[\psi'(\varepsilon_{t1})], \cdots, E[\psi'(\varepsilon_{tp})])$.

- Given $\widehat{\Gamma}_n$ and $\widehat{\Sigma}_n$ consistent estimators of Γ and Σ then

$$\boldsymbol{T}_n = \Upsilon_n^\top \left(\widehat{\Pi}_n A\right)^\top \left(\widehat{\Gamma}_n \widehat{\Sigma}_n^{-1} \widehat{\Gamma}_n\right) \left(\widehat{\Pi}_n A\right) \Upsilon_n \xrightarrow{d} \mathcal{W}_r(p, I),$$

  a standard Wishart distribution with $r = \dim(\mathcal{C}^\perp)$ where

$$\Upsilon_n \Upsilon_n^\top = A^\top \left(\sum_{t=k+1}^{n} \boldsymbol{X}_{t-k} \boldsymbol{X}_{t-k}^\top \right) A.$$

- To test $H_0 : \mathcal{C} = \mathcal{C}_0$, use test statistics based on the eigenvalues of $\boldsymbol{T}_n$:
  - maximum eigenvalue — can analytically evaluate the limiting distribution, albeit painfully (Muirhead, 1982) or via simulation;
  - trace — χ^2 limiting distribution.

- Rate of convergence is faster than for LS:
  - For LS estimation, we have $O_p(n^{-1})$ convergence rate;
  - for M-estimation, we have essentially $O_p(n^{-1/2}n^{-\tau}L(n))$ convergence where τ is the smallest of the real components of the eigenvalue of E and $L(n)$ is a slowly varying function.
- Limiting distribution is independent of E and $\{\Delta_n\}$ — thus we don't need to estimate tail indices!

3. FINAL COMMENTS

- The results can be extended to allow drift and other $I(0)$ terms (including an intercept) in the model.
  - Need only correct for estimation of these additional parameters.
- Asymptotic theory for estimators of $\Phi_1, \cdots, \Phi_{k-1}$ is non-standard — point process asymptotics.
- Open question: Is a "domain of attraction" assumption necessary?
  - Does $\boldsymbol{T}_n \xrightarrow{d} \mathcal{W}_r(p, I)$ if $\boldsymbol{a}^\top \boldsymbol{\varepsilon}_t$ has infinite variance for all non-zero $\boldsymbol{a}$?
- Extensions to domains of attraction with a Normal component also are possible.