Phase Estimation for Fluctuation Processes

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Conference in Honor of Manfred Deistler

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Oscillation Processes

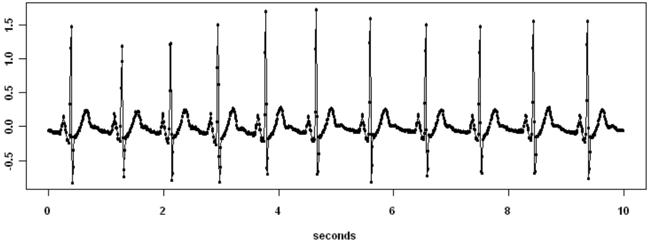

Figure: 1000 Observations from an ECG

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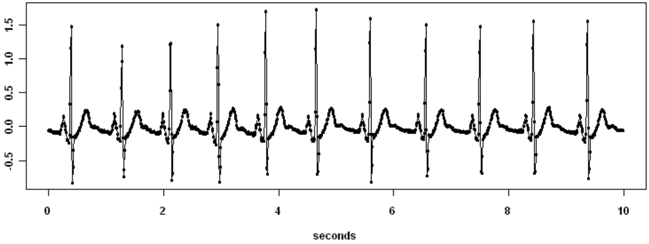

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Idea for a model: $y_t = a_t f(\phi_t) + b_t + \epsilon_t$

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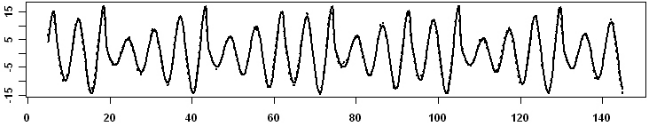

Figure: 1415 Observations from a Rössler Attractor

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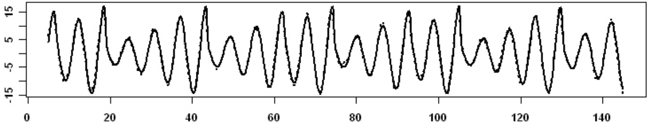

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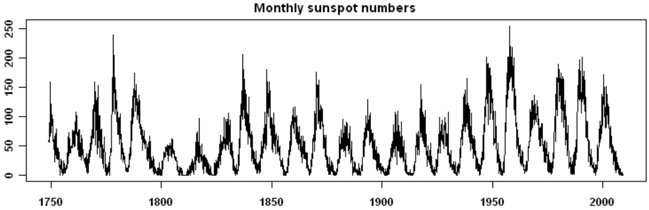

Figure: Monthly sunspot numbers

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- overcome problem of (possibly) negative $\Delta\phi_t$!!

Model for the Phase Process ϕ_t: ACD-model

Model $\Delta\phi_t$ as the durations of an ACD(1,0)-model (Engle and Russell, 1998), i.e.

$$\Delta\phi_t = \tilde{\psi}_t \, \eta_t \quad \text{where} \quad \tilde{\psi}_t = \alpha + \beta \, \Delta\phi_{t-1},$$

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$$\mathbf{E}(\Delta\phi_t) = \frac{\alpha \, \mathbf{E}\eta_t}{1-\beta},$$

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Observation equation

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$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} = A \begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} + \begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix},$$

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with independent $\epsilon_t, \eta_t, \xi_t, \zeta_t$, $\quad (\xi_t, \zeta_t)^T \sim \mathcal{N}(\mathbf{0}, Q)$, $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$.

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$\eta_t \sim \gamma(2, 2)$

$(\psi_t = \tilde{\psi}_t \eta_t$ is an auxiliary state)

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Nonparametric view

Observations:

$$y_t = a_t f(\phi_t) + b_t + \epsilon_t, \quad t = 1, \ldots, T$$

with unknown f, ϕ_t, a_t and b_t.

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- ! Most of the identifiability problems still need to be solved !

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- A fixed lag smoother is applied in order to overcome the problems due to sample depletion

Estimation of parameters (f known, e.g. $f(x) = \cos(x)$)

EM-algorithm:

Maximize and calculate

$$\mathcal{Q}(\theta|\theta^{(m)}) = \mathbf{E}_{\theta^{(m)}}[\log p_\theta(\mathbf{x}_{0:T}, y_{1:T})|y_{1:T}]$$

where

$$x_t := (\phi_t, \psi_t, a_t, b_t)'.$$

Model: $y_t = a_t \cos(\phi_t) + b_t + \epsilon_t, \quad \Delta\phi_t \sim ACD(1,0), \quad T = 1000$

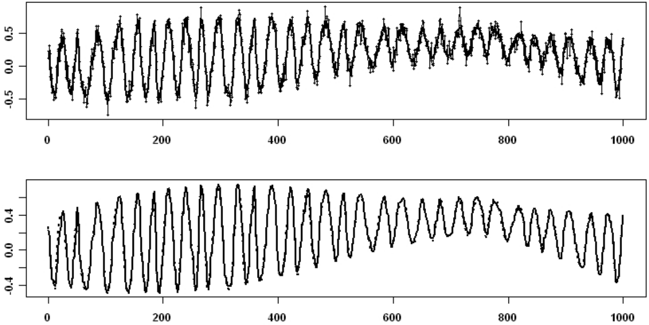

Figure: True and estimated signal, low-noise-case

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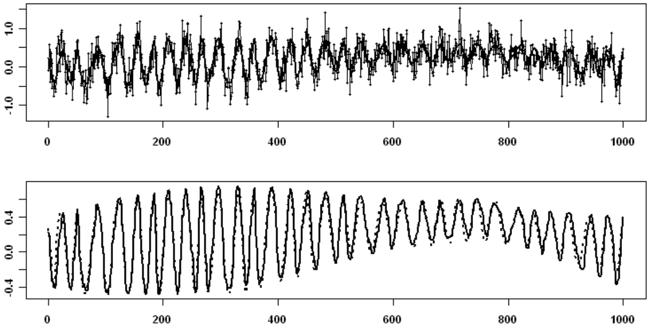

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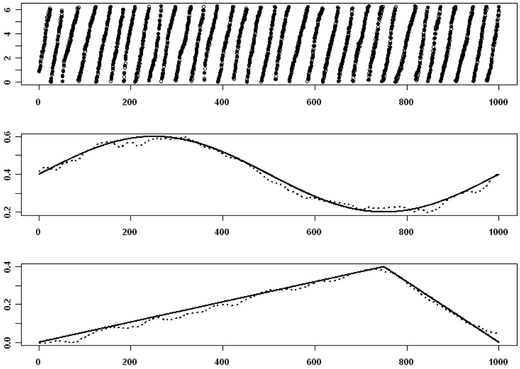

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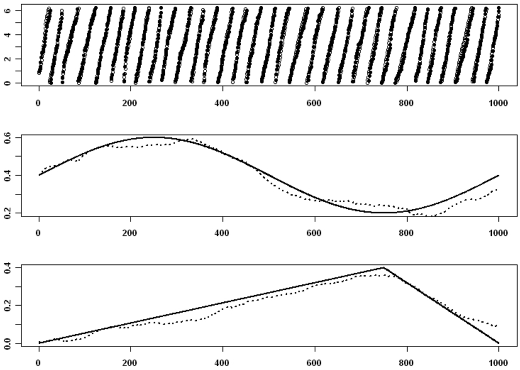

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Noisy Rössler Attractor - low noise

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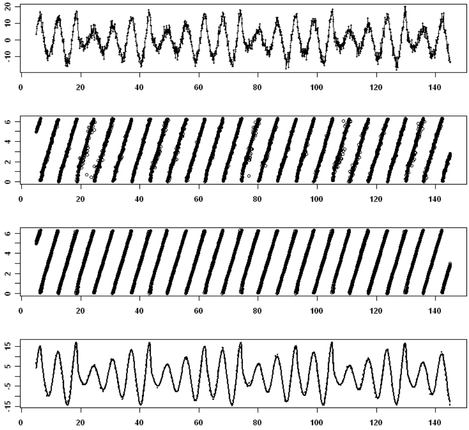

Figure: x_1-component of the Rössler attractor with additive i.i.d. N(0; 4) noise, Hilbert transform + true, Phase estimate + true, estimated signal +true

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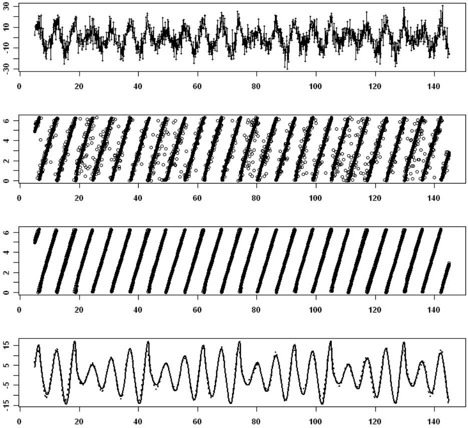

Figure: x_1-component of the Rössler attractor with additive i.i.d. N(0; 40) noise, Hilbert transform + true, Phase estimate + true, estimated signal +true

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- For given f_j predict (the conditional density of) a_t, b_t, ϕ_t by the particles of the particle smoother
- Estimate f (i.e. obtain f_{j+1}) by the Nadaraya-Watson estimate

Example with f unknown - ECG-data

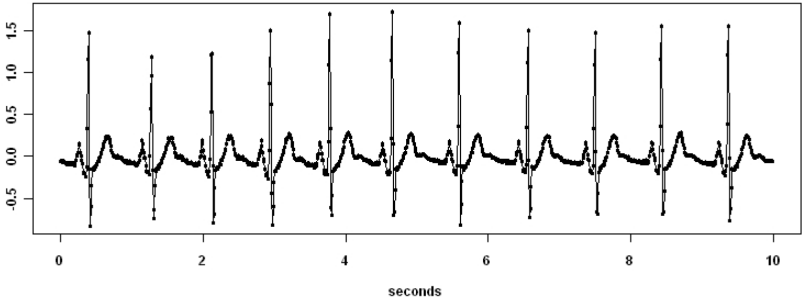

Figure: T=1000 observations of a human ECG

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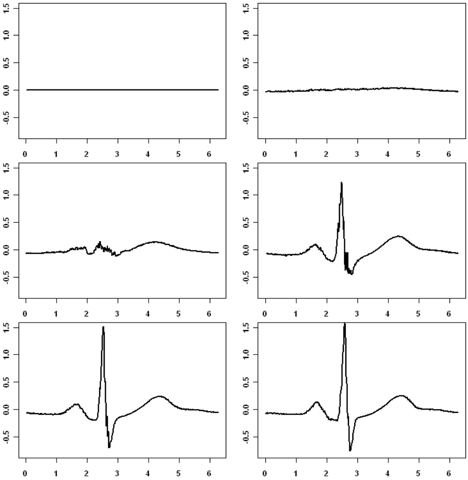

Figure: Estimate of f after each iteration of the EM-algorithm

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